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THE EFFECT OF THE LORENTZ POLARIZATION TERM IN IONOSPHERIC CALCULATIONS

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ABSTRACT. Some calculations are made concerning the propagation of waves through an ionospheric region having a parabolic distribution of electron-density with height. The Lorentz term is included, but the effects of electron collisions and of the earth's magnetic field are neglected. Expressions are deduced for the group path at vertical incidence as a function of wave frequency, and for the horizontal range covered at oblique incidence as a function of angle of incidence and wave frequency. The results are presented graphically and are compared with calculations in which the Lorentz term is neglected. It is shown that experiments in which vertical-incidence and oblique-incidence propagation through region *F* are compared cannot be expected to be accurate enough to decide whether or not the Lorentz term should be included.

§ 1. INTRODUCTION

It is well known that there is at present considerable doubt as to whether the Lorentz polarization term(1) should be included in the expressions for the force acting on a free electron in the ionosphere. If this term is omitted, together with the effects of the earth's magnetic field and of electron collisions, many of the expressions involved in ionosphere theory assume a simple form, and in consequence the term has often been purposely neglected in calculations. One such calculation is that for the case of waves incident upon an ionospheric region having a parabolic distribution of ionization density with height. Several workers(2, 3) have given expressions for the group path for waves incident vertically on such a region. Since calculations of this type have been used in deducing the shape of the ionized regions from the form of the observed *P′-f* curves, it is of importance to know how far the deductions would be modified if the Lorentz term were included. In the first part of this paper, § 3·1 (*a*), expressions are developed which give the phase and group path for a wave incident vertically on a parabolic ionized region for the case in which the Lorentz term is included. The results of the two calculations are compared in § 4 (*a*).

Another calculation which is very considerably simplified by the neglect of the Lorentz term is that in which vertical-incidence propagation through the ionosphere is compared with oblique-incidence propagation. Martyn$^{(4)}$ has shown that if the Lorentz term is neglected the behaviour of waves at oblique incidence can be deduced from that at vertical incidence, whatever the vertical distribution of ionization may be. It has not been found possible to modify this useful general theorem to include the Lorentz term. For the special case of a parabolic region, however, analytical expressions representing the behaviour of waves at oblique incidence, when the Lorentz term is included, are deduced in § 3·1 (*b*). By comparison of the vertical-incidence and oblique-incidence results, for the special case of the parabolic region, we can obtain some idea as to how nearly Martyn's theorem continues to be true when the Lorentz term is included.

Under certain circumstances it might be expected that the oblique-incidence behaviour deduced from an observed vertical-incidence P'-f curve would be different according as the Lorentz term is, or is not, included. If this were the case it would be possible to decide by experiment whether the term should be included, by making simultaneous observations of P'-f curves at vertical and at oblique incidence. Accurate experiments of this type, carried out over a distance of 464 km., have recently been described by Farmer, Childs and Cowie$^{(5)}$ and in § 4 (*b*) their results are examined in detail to see whether they can lead to a decision about the Lorentz-term ambiguity. It is concluded that no such decision can be reached by means of experiments of this type.

Throughout this paper the effects of the earth's magnetic field and of electron collisions have been neglected, since it does not appear possible to include them in generalized analytical expressions. In order to estimate the change due to the inclusion of these effects, numerical computations would have to be made for any special case.

§ 2. SOME GENERAL CONSIDERATIONS

The following is a list of the symbols used in this paper: f, wave frequency; l, a quantity equal to $\frac{1}{3}$ for a theory which includes the Lorentz term and 0 for a theory which omits it; N, electron-density; $P(f, \phi)$, optical path of a wave of frequency f incident on the layer at an angle ϕ; $P'(f, \phi)$, group path of the same wave; p, angular frequency of the wave; p_c, angular frequency of a wave which just penetrates the layer at vertical incidence; $p_0^2 = 4\pi Ne^2/m$, where e, m represent the charge and mass of an electron; $X(f, \phi)$, horizontal range covered, at the bottom of the layer, by a wave of frequency f and angle of incidence ϕ; X_T, total range as measured at ground level; z, distance below the level of maximum ionization; z_0, distance from ground to bottom of layer; $\epsilon = p/p_c$; ϕ, angle of incidence on the ionized region; μ, refractive index.

In what follows we shall represent the Lorentz term by l, and shall refer to a theory in which the Lorentz term is neglected as "the case in which $l=0$", and a theory in which it is included as "the case in which $l=\frac{1}{3}$".

Martyn$^{(4)}$ has stated a theorem which is of great use in relating vertical and oblique propagation conditions for waves travelling in a stratified ionosphere with any vertical distribution of ionization in the case in which $l=0$. He assumes that the refractive index is given in terms of the electron-density and the wave-frequency by an expression of the form

$$1-\mu^2=F(N)/f^2,$$

from which it follows that the group velocity is $c\mu$. This form of expression neglects the Lorentz term and the effect of electronic collisions and of the earth's magnetic field. It is further assumed that the ionosphere is horizontally stratified so that N is a function of z only.* It is then shown that the equivalent path $P'(f, \phi)$ for propagation with an angle of incidence ϕ for a wave frequency f is related to the equivalent path at vertical incidence, for a frequency $f \cos \phi$, by the expression

$$\cos\phi \, P'(f, \phi)=P'(f\cos\phi, 0). \qquad \text{......(1)}$$

We shall call this relation Martyn's theorem.†

Under the assumptions mentioned above, it was also shown by Breit and Tuve(8) that the horizontal range $X(\phi)$, covered by a wave incident on the ionosphere at an angle ϕ, before it returns again to the ground, is given by

$$X(\phi)=\sin\phi \, . \, P'(\phi), \qquad \text{......(2)}$$

under the assumptions mentioned above. This relation we shall call Breit and Tuve's theorem. A combination of (1) and (2) gives

$$X(\phi, f)=\tan\phi \, . \, P'(f\cos\phi, 0), \qquad \text{......(3)}$$

relating the range at any angle and frequency, to the normal-incidence P'-f curve.

§ 3. CALCULATIONS FOR A PARABOLIC DISTRIBUTION OF IONIZATION DENSITY

We assume a parabolic ionization density measured in terms of p_0^2 (equal to $4\pi Ne^2/m$) and we measure distance z downwards from the level of maximum ionization, and arrange our scale of height so that the half thickness of the layer is unity and

$$p_0^2=\pi_0^2(1-z^2).$$

We further assume a region of thickness z_0, between the ground and the lower edge of the layer, in which $\mu=1$.

3·1. *The case in which* $l=\frac{1}{3}$. Omitting the effect of electron collision and of a superimposed magnetic field, but including the Lorentz term we have(9)

$$\mu^2=\frac{1-2p_0^2/3p^2}{1+p_0^2/3p^2}=2\,\frac{z^2-(1-3p^2/2\pi_0^2)}{(1+3p^2/\pi_0^2)-z^2}. \qquad \text{......(4)}$$

If we now write p_e for the angular frequency which makes μ equal to zero when z is zero, i.e. the vertical incidence penetration frequency, we have

$$p_e^2=\tfrac{2}{3}\pi_0^2$$

and, if we write

$$p/p_e=\epsilon,$$

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* This assumption neglects the curvature of the earth, the effect of which has been investigated by Millington(6) and by Newbern Smith(7).

† Martyn(4) did not publish his theorem quite in the general form given here but he clearly recognized the implications of the form he gave.

equation (4) becomes

$$\mu^2 = 2\,\frac{z^2-(1-\epsilon^2)}{(1+2\epsilon^2)-z^2} = 2\,\frac{z^2-a^2}{b^2-z^2}, \qquad \text{......(5)}$$

with

$$a^2 = 1-\epsilon^2, \qquad \text{......(6)}$$

$$b^2 = 1+2\epsilon^2. \qquad \text{......(7)}$$

(*a*) *Normal incidence.* We now calculate the phase path $P(\epsilon, 0)$ in the layer alone for normal incidence and a frequency corresponding to ϵ. We have

$$\begin{aligned} P(\epsilon, 0) &= 2\int_{\mu=1}^{\mu=0} \mu\, dz \\ &= 2\sqrt{2}\int_{z=1}^{z=0} \sqrt{\frac{z^2-a^2}{b^2-z^2}}\, dz \\ &= 2\sqrt{2}\, bk^2 \Big[F(\alpha_0, \theta) - D(\alpha_0, \theta)\Big]_{\theta=0}^{\theta=\pi/2}, \qquad \text{......(8)} \end{aligned}$$

where

$$\sin^2\alpha_0 = k^2 = \frac{b^2-a^2}{b^2} = \frac{3\epsilon^2}{1+2\epsilon^2}, \qquad \text{......(9)}$$

$$\theta = \sin^{-1}\sqrt{\tfrac{2}{3}} = 54{\cdot}8^\circ,$$

and $F(\alpha, \theta)$ and $D(\alpha, \theta)$ are the elliptic integral functions defined on pp. 124 and 128 of Jahnke and Emde's *Functionen Tafeln* (Second Edition, 1933).

We now deduce the group path $P'(\epsilon, 0)$ by using the relation*

$$P' = P + \epsilon\,\frac{dP}{d\epsilon}. \qquad \text{......(10)}$$

This involves a differentiation of the elliptic integrals with respect to ϵ, but since ϵ enters only through the angle α, and not into the limits of integration, this differentiation can be carried out by using the expressions given on p. 129 of Jahnke and Emde's *Tafeln*. After considerable algebraic manipulation we arrive at the expression

$$P'(\epsilon, 0) = 2\sqrt{2}\Big[(1+2\epsilon^2)^{-\frac{1}{2}}\{(1+4\epsilon^2)\,E(\alpha_0, \theta) - (1-2\epsilon^2)\,F(\alpha_0, \theta)\} \\ + 3\epsilon^2(1+2\epsilon^2)^{-\frac{1}{2}}\,G(\alpha_0, \theta)\Big]_{\theta=\theta_0}^{\theta=\pi/2},$$

where

$$G = \frac{\sin\theta\cos\theta}{\sqrt{(1-k^2\sin^2\theta)}},$$

and this finally gives

$$P'(\epsilon, 0) = 2\sqrt{2}\,(1+2\epsilon^2)^{-\frac{1}{2}}\Big[(1+4\epsilon^2)\,E(\alpha_0, \theta) - (1-2\epsilon^2)\,F(\alpha_0, \theta)\Big]_{\theta=\theta_0}^{\theta=\pi/2} - \tfrac{4}{3}k^2. \qquad \text{......(11)}$$

The total group path is now obtained by adding $2z_0$ to this quantity. A graph showing $\frac{1}{2}P'(\epsilon, 0)$ as a function of ϵ, as calculated from equation (11), is shown in figure 1, curve *a*.

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* This relation results⁽¹²⁾ from the general theory of group velocity, and does not depend on any special assumptions about the form of the expression for μ.

(*b*) *Oblique incidence.* We shall find that it is most convenient to plot curves which show the range $X(\epsilon, \phi)$ in the layer for an angle of incidence ϕ and a frequency ϵ. This is given by

$$X(\epsilon, \phi) = 2\int_{\mu=1}^{\mu=s} dx = 2s\int_{\mu=1}^{\mu=s} \frac{dz}{\sqrt{(\mu^2 - s^2)}}, \qquad \text{......(12)}$$

where s is written for $\sin\phi$.

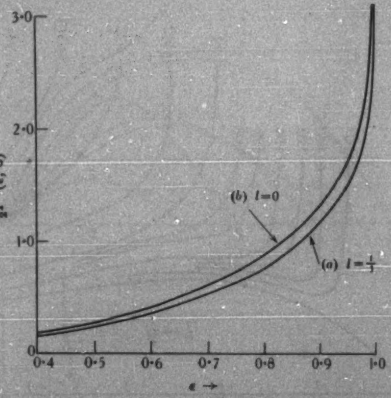

Figure 1. Equivalent height $\frac{1}{2}P'(\epsilon, 0)$ for vertical incidence as a function of $\epsilon = f/f_c$. Curve *a* corresponds to $l=\frac{1}{3}$ and curve *b* to $l=0$.

From (5) we have
$$\mu^2 - s^2 = (2+s^2)\frac{z^2 - c^2}{b^2 - z^2}, \qquad \text{......(13)}$$
where
$$c^2 = (2a^2 + s^2b^2)/(2+s^2); \qquad \text{......(14)}$$
so that (12) becomes

$$\begin{aligned} X(\epsilon, \phi) &= 2s(2+s^2)^{-\frac{1}{2}}\int_{z=1}^{z=c}\sqrt{\frac{b^2-z^2}{z^2-c^2}}\,dz \\ &= 2s(2+s^2)^{-\frac{1}{2}}(1+2\epsilon^2)^{\frac{1}{2}}\Big[E(\alpha_1, \theta) - F(\alpha_1, \theta)\Big]_{\theta=\theta_1}^{\theta=\pi/2}, \qquad \text{......(15)} \end{aligned}$$

where
$$\sin^2\alpha_1 = \frac{6}{2+s^2}\,\frac{\epsilon^2}{1+2\epsilon^2},$$
$$\sin^2\theta_1 = \tfrac{1}{3}(2+s^2).$$

The total range X_T, including the range in the region in which $\mu=1$, is given by adding $2z_0\tan\phi$ to expression (15). Graphs of $\frac{1}{2}X$ as a function of ϵ and ϕ as calculated from expression (15) are shown in figure 2*a*. In drawing these graphs it is helpful to notice that the expression (15) becomes infinite where $\alpha_1 = \pi/2$, i.e. where $\sin^2\phi = 2(\epsilon^2 - 1)/(2\epsilon^2 + 1)$.

3·2. *The case in which* $l=0$. If the Lorentz term is not included we have

$$\mu^2 = 1 - \frac{\pi_0^2}{p^2}(1 - z^2)$$

and the vertical-incidence penetration frequency p_c is equal to π_0, so that

$$\mu^2 = 1 - \epsilon^2 (1 - z^2). \qquad \text{......(16)}$$

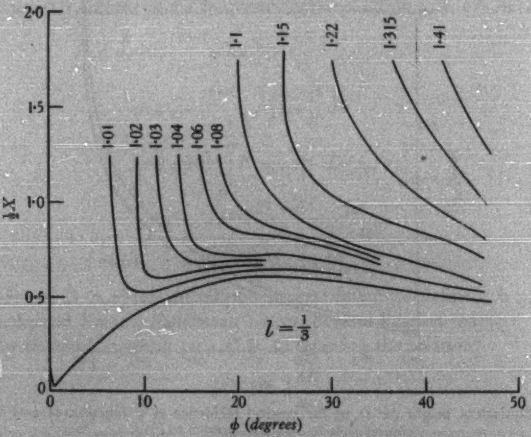

Figure 2*a*.

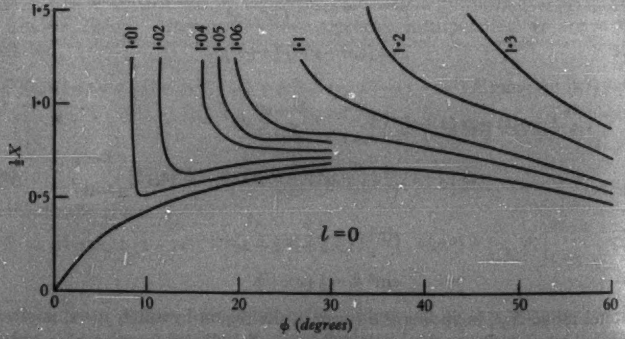

Figure 2*b*.

Figure 2. Half range in the region as a function of angle of incidence for different values of frequency. The numbers against the curves represent the frequencies in terms of $\epsilon = f/f_c$. Figure 2*a* corresponds to $l=\frac{1}{3}$ and figure 2*b* to $l=0$.

Previous workers[2,3] have shown that the group path in the layer at vertical incidence is then given by

$$P'(\epsilon, 0) = \epsilon \log_e \left(\frac{1+\epsilon}{1-\epsilon}\right). \qquad \text{......(17)}$$

A graph showing $\frac{1}{2}P'$ as a function of ϵ is given in figure 1, curve (*b*).

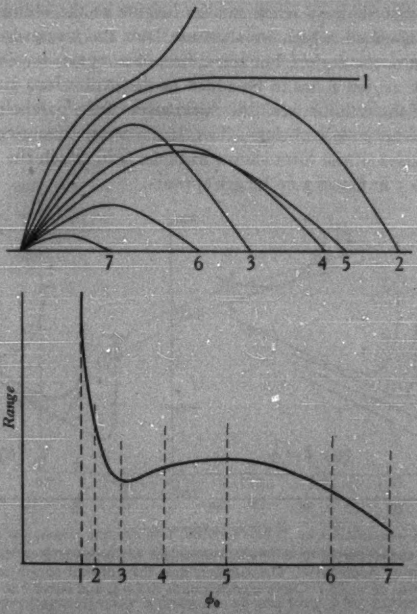

Figure 3. Explanation of {range, angle} curves.

To find the range in the layer for any frequency we combine equations (17) and (3) to obtain

$$X(\epsilon, \phi) = \epsilon \sin\phi \log_e \left(\frac{1+\epsilon\cos\phi}{1-\epsilon\cos\phi}\right). \qquad \text{......(18)}$$

The range becomes infinite when $\sec\phi = \epsilon$. Graphs of $\frac{1}{2}X(\epsilon, \phi)$ as a function of ϵ and ϕ, calculated from equation (18) are shown in figure 2*b*.

3·3. *The {range, angle} curves.* It is of interest to consider the form of the {range, angle} curves shown in figure 2. In interpreting these curves we must remember that they apply to the range of a wave which starts *in the layer* at the place where $\mu = 1$. For a fixed frequency, greater than the vertical-incidence penetration frequency (so that $\epsilon > 1$), the wave penetrates the layer for all angles less than the limiting angle marked 1 in figure 3. At the angle marked 1 the ray just becomes horizontal at the level of maximum ionization, and since, at that point, the vertical gradient of μ is zero, it remains horizontal. As the angle of incidence is increased the range rapidly drops to a minimum corresponding to the angle 3, and then rises to a maximum at 5 after which it decreases steadily to zero as the angle is increased to $\pi/2$. The decrease in the range for angles greater than 5 is due to the fact that the rays reach smaller heights as the obliquity is increased. In the model ionosphere which we shall use later the waves do not start at the bottom of the parabolic region, but travel first in a region devoid of electrons, so that the quantity $2z_0 \tan\phi$ has to be added to the expressions given in equations (15) and (17). If the ratio between the thicknesses of the parabolic region and the thickness of the empty region below is less than a certain quantity, then we obtain {range, angle} curves of the form shown in figure 4 in which the maximum range corresponding to 5 in figure 3 no longer appears.

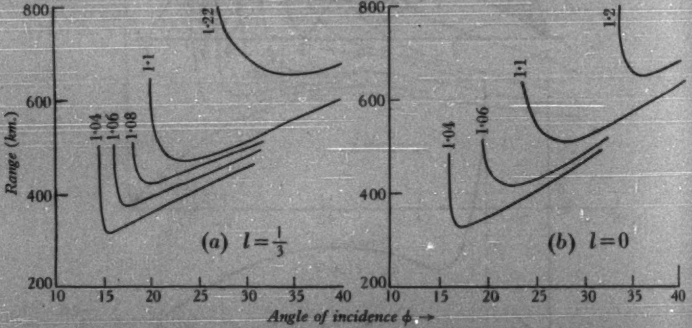

Figure 4. {Range, angle} curves at different frequencies corresponding to the vertical-incidence P'-f curve given by Appleton(3). The numbers near the different curves represent frequencies measured in terms of $\epsilon = f/f_e$. Curve *a* corresponds to $l=\frac{1}{3}$, and curve *b* to $l=0$.

§4. DISCUSSION OF RESULTS

(*a*) *Vertical incidence.* The P'-f curves for vertical incidence, for the two cases in which $l=0$ and $l=\frac{1}{3}$, are shown in figure 1 and are seen to be of approximately the same form. Theoretical P'-f curves of this type (for the case in which $l=0$) have been used by several workers[3, 11] to deduce the approximate shape of a reflecting region from the observed form of the P'-f curve. The present results show that there would be no important change in the deductions if the Lorentz term were included. It is very probable that the change produced by including the effect of the earth's magnetic field would be at least as large.

(*b*) *Oblique incidence.* It has previously been pointed out[10] that if an ionospheric region existed with a sharp lower boundary, and a uniform horizontal distribution, then by comparing the penetration frequencies for vertical and for oblique incidence it should be possible to decide whether $l=0$ or $l=\frac{1}{3}$. Attempts to make experiments of this kind by the use of region E showed that when it had a sharp enough boundary its structure was very variable in a horizontal direction. Recently Farmer, Childs and Cowie(5) have described experiments in which they compared the behaviour of waves reflected from region F and received simultaneously at vertical and at oblique incidence. They found that this region, unlike region E, was often sufficiently uniform for accurate results to be obtained, and it was hoped that some information about the Lorentz term might be deducible from their experiments. This possibility will next be discussed, on the assumption that region F is approximately parabolic in form.

The experimental results of Farmer, Childs and Cowie consisted of simultaneous observations of vertical-incidence P'-f curves and of the highest frequency which could be propagated over a distance of 464 km. This highest frequency was called the limiting frequency. We shall next show how the expected limiting frequency can be deduced theoretically from the vertical-incidence P'-f curve in the two cases $l=0$ and $l=\frac{1}{3}$.

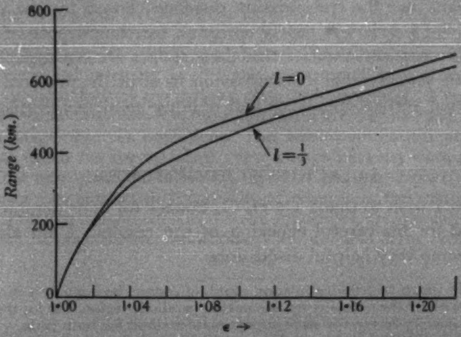

Figure 5. Limiting frequency, expressed in terms of $\epsilon=f/f_0$, as a function of horizontal range, for the case of the P'-f curve referred to in figure 4.

The theoretical P'-f curves of figure 1 (a, b) are first fitted to the observed vertical-incidence P'-f curve and, from the adjustment required, the form and the height above the ground of the (assumed) parabolic region is determined for each of the two cases $l=0$ and $l=\frac{1}{3}$. By using the curves of figure 2 it is then possible to deduce, for both cases, a set of curves showing the half-range $\frac{1}{2}X$ as a function of the angle of incidence ϕ for a series of frequencies $\epsilon=p/p_0$. As a representative example of region-F observations we shall take the case discussed in Prof. Appleton's Bakerian lecture(3) and represented in his figure 8. The {range, angle} curves deduced for the two cases are shown in figure 4. From these curves it is next possible to deduce the highest limiting frequency which will traverse a given horizontal range, and the relation between range and limiting frequency is shown in figure 5.*

From figure 5 it appears that, for large and for small ranges, the predicted limiting frequency is the same in the two cases $l=0$ and $l=\frac{1}{3}$, but for ranges round about 500 km. the predicted frequencies may differ by as much as 2 per cent. Although this difference is small it might have been significant in the case of the very accurate experiment of Farmer, Childs and Cowie, if the form of region *F* in their case had been similar to that which we have been considering. A detailed examination, by the methods outlined above, of all the *P′-f* curves used by Farmer, Childs and Cowie† has shown, however, that in their case the difference between the deduction from the assumption that $l=0$ and that from the assumption that $l=\frac{1}{3}$ was only about 1 per cent, so that their experiments, with an accuracy of 0·5 per cent, cannot be said to have any significance in this connexion.‡ From figure 5 it appears that the transmission distance chosen for the experiments of Farmer, Childs and Cowie was nearly optimum for the purpose of differentiating between the two cases, and the fact that their results are not capable of providing this differentiation suggests that it is impossible to settle the problem of the Lorentz term by comparing vertical-incidence and oblique-incidence propagation through region *F*.

§ 5. ACKNOWLEDGEMENTS

In conclusion I wish to thank Dr H. G. Booker for considerable help with the mathematics and for his careful checking of the results. I am also indebted to Dr Farmer for some very helpful discussions.

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* For the case in which $l=0$ the calculation could, of course, be done by a direct application of Martyn's theorem, without the necessity for assuming a parabolic region, but for the present purpose it seemed more satisfactory to use the same approximate method for both cases.

† I am very much indebted to Messrs Farmer, Childs and Cowie for putting the details of all their experimental results at my disposal for the purpose of this examination.

‡ It would be expected that the earth's magnetic field would produce effects at least as large as those due to the inclusion of the Lorentz term.

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THE RELATIVE LUMINOSITY OF RADIATION FOR THE AVERAGE PHOTOMETRIC OBSERVER (2)

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ABSTRACT. This paper describes measurements of the mean relative luminosity of radiation at five wave-lengths in the spectrum, made by 20 observers. Work described by the author in a previous paper on the relative luminosity at wave-lengths 5461 and 5780 A. of the mercury spectrum is followed up by the inclusion of the sodium doublet at 5893 A. and the cadmium lines at 5086 and 6439 A. Electric discharge lamps were used as light-sources, and their visual intensity in different spectrum lines was compared with the energy radiated in these lines by the method described previously. A connexion between the values for individual observers and their Y/B ratios is confirmed and utilized to reduce the results to average-observer conditions. The results are consistent with a relative-luminosity curve which fits the standard curve except between wave-lengths 5550 and 6350 A., where the new observed values are up to 11 per cent higher than the standard values. The results also suggest the possibility of changes in the values obtained over a period of time. Possible causes, such as dietetic influences, are discussed, as are also the effects of changes in photometric conditions. The probable error of the results is also dealt with.

§ 1. INTRODUCTION

A previous paper(1) by the present author describes measurements on the relative visual luminosity of radiation of wave-lengths 5780 A. and 5461 A. by 16 observers. While such measurements may give some information about the position in the spectrum of the luminosity maximum (which lies between these wave-lengths), they give little about the general shape of the relative-luminosity curve. In the paper referred to, therefore, there was mentioned the possibility of extending the scope to embrace other wave-lengths. The present paper describes progress on these lines, and gives summaries of the original and later measurements on the two mercury lines. The method adopted and apparatus used was the same throughout. Photometric comparisons of the relative luminous intensity of different monochromatic radiations were made with a Guild flicker photometer(2), along with comparative measurements of the energies of the radiations using a calibrated photocell of the Elster-Geitel type. Reference is therefore made to the previous paper for details concerning the principles of the method, the calibration of the photocell, and the testing of the thermopile which formed the ultimate basis of the energy measurements.

In view of the interesting connexion between the individual luminosity ratios and Y/B ratios(3) found in the first investigation, a new and complete set of determinations of Y/B ratios was undertaken in connexion with the present work. The usefulness of these data will be explained later.

§ 2. THE LIGHT SOURCES

For wave-lengths 5780 and 5461 A. a normal 250-W. mercury discharge lamp was used. For wave-length 5893 A. a commercial type of 150-W. sodium lamp proved suitable. The search for powerful sources of other spectral types proved less fruitful than was at first expected. The cadmium lines at 5086 and 6439 A. seemed likely to be useful, but the usual type of colour-corrected cadmium-mercury discharge lamp did not provide sufficient energy, in these lines, for the method in view. However, through the courtesy of the Research Laboratories of The British Thomson Houston Co., Ltd., three special 400-W. discharge lamps containing a considerable excess of cadmium were made available for the work. Under laboratory conditions, these lamps could be over-run to about 480 W. each (on d.c.) and at this wattage emitted the desired lines at considerable strength. After due allowance had been made for the absorption of the selective filters to be used, it was found that two such lamps, side by side, each operating at 480 W., sufficed for the 5c86-A. line. For the 6439 A. line, of considerably lower luminosity, a mirror was used behind the same pair of lamps at such an angle as to bring the images of the lamps apparently alongside the lamps themselves, as seen from the photometer.

These same lamps, although containing mercury, were not used for working with the mercury spectrum in view of the difficulty of satisfactorily eliminating, by means of filters of reasonable density, the unwanted cadmium spectrum.

There appear to be no other sources of monochromatic light giving other wave-lengths suitable for the method used, and also steady in output and convenient to control.

§ 3. THE COLOUR FILTERS

The filters used were: (1) for mercury 5461 A., 10 mm. of Chance didymium glass with 2·78 mm. of Chance-Parsons green glass no. 5; (2) for mercury 5780 A., 3·68 mm. of Chance-Parsons deep orange glass no. 3 with 1 cm. of 0·3-M.* aqueous solution of cupric chloride; (3) for sodium 5893 A., no filter; (4) for cadmium 5086 A., Wratten gelatine no. 75 with 1 cm. of 0·2-M. aqueous solution of potassium chromate; (5) for cadmium 6439 A., 4·50 mm. of Chance-Parsons selenium red glass no. 2.

The spectral purity of the transmitted radiation was in each case examined by means of a monochromator and of the calibrated photocell used for the subsequent work. Calculations were made of the impurity expressed as a fraction of the *visual luminosity* of the wanted spectrum line, and also in terms of the *photocell output* due to impurity as compared with that for the same line. The latter figure may of course be greater or less than the visual impurity, in dependence on the spectral sensitivity of the cell, but the error in the final results due to spectral impurity will be the difference between the percentage impurities expressed in these two ways. This difference was in no case found to exceed about 1 per cent. Possible errors due to the presence of infra-red lines or background, especially in the case of the sodium spectrum, were avoided by the use of a photocell with no response in this region.*

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* M. stands for the molecular weight in grams per litre of solution.

Mention must here be made of the fact that during the examination of the filters for the purpose of the present investigation, evidence of diffusion in the Chance-Parsons deep orange glass was noticed and led to an examination of the previous results in so far as this diffusion may have affected the relative readings of the visual and photoelectric photometers. As a consequence a correction of 1 per cent was found necessary in the case of the earlier set of values of K_{5780}/K_{5461} and has been applied in the tabulated results.

All the filter components were optically worked, and care was taken to avoid parallelism of the components which might have led to errors due to the production of interference patterns on the photometer.†

§4. SOME PRELIMINARY NOTES ON THE METHOD AND RESULTS

As regards the method it is to be noted that the mercury and cadmium sources were operated throughout on direct current to avoid the slight inconvenience of stroboscopic effects in the flicker photometer. The sodium lamp, however, was necessarily operated on an alternating supply, being unsuitable in design for direct-current operation. As on the previous occasion, the photoelectric photometer used for the energy-measurements was designed to have a sufficiently long time constant in the input circuit to ensure correct integration of the intermittent output of light from the sodium lamp.

The calibration of the photocell against a linear thermopile was carried out on a spectrophotometer set-up, with a tungsten lamp as light-source, and also checked in a similar set-up, with the discharge lamps as light-sources, in case of any possible residual wave-length or stray light errors in the first set-up. Good agreement was obtained between the two sets of measurements.

In the photometric system used in the luminosity-determinations, the flicker photometer and the photoelectric photometer were arranged on a slide so that the test surface of either could be set in the same plane, in the same position, on the axis of the bench, the selective filter remaining in a fixed position on this axis. In the first investigation the photometers were fixed side by side, the filter being arranged to slide in front of either. This modification, though a refinement, enabled the screening arrangements to be improved, and ensured that both photometers should receive light emitted from the sources in precisely the same direction. The test surface of the Guild photometer was freshly smoked with magnesium oxide, which is known to be non-selective within the required limits.

* In this respect direct measurement of the radiation by a thermoelement is less convenient, in practice.

† Attention was drawn to this possibility by the author's colleague, Dr W. S. Stiles.

A note is now necessary as to the chronological order of the results. The first determination, made with two mercury wave-lengths only, was carried out in July, 1937.(1) The second determination, in which the sodium and cadmium lines also were used, was made in March, 1938. The relative luminosity for the mercury wave-lengths on this occasion was noticeably different from that previously obtained. A third determination of this figure was therefore thought desirable, and was made in October, 1938. This accounts for the inclusion of three columns of figures under wave-length 5780 A. in table 1. The differences exhibited by these three sets of figures are discussed below.

Table 1

A gives tabulated observed values of Y/B ratio, and of relative luminosities K_R calculated for convenience on the luminosity at $\lambda = 5461$ A. being taken as unity. B gives the means of the columns of the figures in A, which represent, owing to unavoidable circumstances, the results obtained by varying groups of observers. C gives mean values of Y/B ratio for these varying groups of observers, and also the mean value for the complete set of 20.

	Y/B ratio	Relative luminosity K_R, K_{5461} being taken as 1					
		5086 A.	5780 A.			5893 A.	6439 A.
		March 1938	July 1937	March 1938	Oct. 1938	March 1938	March 1938
A. Observer							
FMH	1·007	0·508	0·853	0·943	0·905	0·817	0·150
GEVL	1·007	0·500	0·904	0·945	0·913	0·829	0·148
HB	1·084	—	0·911	—	—	—	—
LMcD	0·952	0·498	0·851	0·903	0·892	0·778	0·126
WB	1·048	0·552	0·947	0·982	0·944	0·863	0·156
JSP	0·951	0·502	0·886	0·894	0·889	0·769	0·124
HFM	0·953	0·511	0·842	0·916	0·884	0·782	0·123
JWTW	1·058	0·482	0·943	1·003	—	0·904	0·158
BHC	0·949	0·465	0·903	0·927	0·897	0·787	0·129
THH	1·076	—	1·010	—	—	—	—
FJCB	1·062	0·349	0·938	0·993	0·972	0·868	0·154
HRS	0·950	0·614	0·897	0·916	—	0·824	0·119
GCC	0·996	—	0·928	—	—	—	—
CD	0·948	0·505	0·874	0·915	0·879	0·785	0·108
GWG-S	1·130	0·460	1·007	1·063	1·023	0·963	0·188
BJO	1·002	0·471	0·899	0·965	—	0·853	0·141
BFD	1·069	0·465	—	0·967	—	0·876	0·158
CJWG	0·985	0·411	—	0·930	0·924	0·804	0·117
LJC	1·067	0·420	—	0·989	—	0·898	0·165
GPB	1·121	0·339	—	1·017	—	0·936	0·168
B. Mean K_R	—	0·474	0·912	0·957	0·920	0·843	0·143
C. Mean Y/B ratio of group	1·021	1·015	1·011	1·015	0·999	1·015	1·015
D. Change in K_R per +1 per cent in Y/B	—	−0·007	+0·010	+0·010	+0·010	+0·010	+0·0036
E. Mean K_R reduced to observer whose Y/B = 1·021	—	0·470	0·922	0·963	0·942	0·849	0·145
			Mean 0·942				
F. 0·984 K_R (for Y/B = 1·021)	—	0·462	—	0·927	—	0·835	0·143
G. C.I.E. standard values(4)	—	0·475	—	0·889	—	0·765	0·146

§ 5. RESULTS

The observed results are summarized in table 1. The Y/B ratios of the observers as given are the results of a new determination (1938). The solutions used gave a calculated Y/B ratio of 0·979 based on spectro-photometric measurements, the C.I.E. standard luminosity data being used.

The photometric conditions were as follows: field-diameter, 2°; surround field white, 10° diameter; field-brightness 20 to 35 equivalent metre candles; eye pupil unrestricted; no optical system between eye and photometer surface; photometric test surface, magnesium oxide.

The calculation in lines D and E is an attempt to systematize the figures of line B to represent as nearly as is possible the results which would have been obtained had the whole 20 observers taken part in each set of measurements. A roughly linear relation, discussed in § 6 below at (i), is observed between the individual values of K_R in each column of A and the corresponding Y/B ratio. This calculation is arbitrary, in so far as the relation is not strict, but it enables an adjustment of the correct sign and order to be made.

In D the gradients of the linear relations just mentioned are determined graphically and expressed as change in K_R per 1-per-cent positive change in Y/B ratio. In E the figures in line B are, from these gradients, adjusted to correspond to a Y/B ratio of 1·021 instead of to the values given in line C. A mean is taken for the three determinations for $\lambda = 5780$ A.

In F the results of line E are reduced for comparative purposes so as to be relative to a luminosity value of 0·984 at $\lambda = 5461$ A. (interpolated standard C.I.E. value) instead of to a basis of unity at this wave-length. This gives observed values directly comparable with G, the values of luminosity interpolated from the C.I.E. standard table.

§ 6. DISCUSSION OF RESULTS

(i) *Relative luminosity, and ratios Y/B.* In the previous paper, dealing only with the ratio K_{5780}/K_{5461}, a relation between this ratio and the Y/B ratio of the observer was pointed out. The present results again show such a relation, not only for the values of K_R for 5780 A., but also for those for 5893 and 6439 A. This may be seen by plotting individual values of K_R for each of these wave-lengths against corresponding Y/B ratios. For each wave-length, the straight line best representing the results has a different slope, represented by line D in table 1. In the case of wave-length 5086 A. such a process yields a system of points distributed much more irregularly. This is to be expected from the known fact that any group of individual luminosity curves plotted in the usual way shows greater differences in the blue region than elsewhere. However, the slope of the best straight line is here negative, as may be expected, and the approximate value is given in line D of table 1, where the application of the slope figures has already been explained.

(ii) *Changes in relative luminosity.* Examination of the three columns under 5780 A. in section A of the table shows that the March readings are systematically higher than the July and October readings, while in the majority of cases the values fall in the order March, October, July. The differences in the case of any one individual observer are not always beyond the limits of experimental error, but if we take the corrected mean values (E in table 1) there does seem to be a certain amount of evidence that for the whole group of observers there are real differences between the mean values of K_R determined on the three different occasions.

Such differences may result either from systematic errors associated with the experimental equipment, or from some systematic physiological change in the group of eyes represented. The former possibility has been followed up by the author and has led only to the correction of the first set of values, necessitated by diffusion by a filter, and which is mentioned in § 3. Differences in photocell calibrations would indeed introduce systematic differences in the three sets of results for 5780 A. These are not to be discounted, but the estimated errors likely to arise from this source are considerably less than the range 0·922 to 0·963 shown by the figures at E in table 1. The hypothesis of a physiological change is attractive, though it must be emphasized at once that any experimental proof of the influence of physiological factors would almost certainly be attended by difficulties. In particular, the task of ascertaining which physiological factor was responsible for any specific variation in visual spectral response would almost certainly involve a considerable amount of careful research.

There is, however, a growing volume of evidence that vitamins play an important part in vision—in particular as regards the influence of vitamin A on retinal adaptation(5). It is interesting therefore to speculate as to the influence of vitamin-supply on the curve of relative spectral luminosity for the human eye. Line E in table 1 shows a value 0·922 ($\lambda = 5780$ A.) obtained just after midsummer, when the supply of vitamin A may reasonably be expected to be near the maximum. At a later stage of the year (October) the corresponding mean relative luminosity is 0·942, while in March, when the supply of vitamin A may be near a minimum, it has risen to 0·963. This suggests, without however affording any definite proof, that the eye becomes relatively more sensitive to yellow or red as it becomes deficient in vitamin A. Such a suggestion is interesting when considered in relation to the work of other investigators such as Arndt(6), Dresler(7), Federov and Federova(8), Jainski(9) and König(10), all of whom obtain luminosity curves which are relatively higher in the yellow region (and in some cases in the red also) than the standard curve. It is also consistent with the assumption that (caeteris paribus) the observers concerned in the American investigations on which the standard curve is based were not deficient in vitamin A. As has been indicated above, however, definite analytical proof of such a possible effect as this is not yet available.

(iii) *The observed values of relative luminosity.* In the absence of such proof, and in order to summarize the present measurements, the mean value of 0·942 is taken for the luminosity at 5780 A. relative to unity at 5461 A. The values are then reduced by the factor 0·984 (the tabulated value at 5461 A. for the standard curve), giving the values in line F in table 1. These reduced values are then comparable with the standard values (line G) assuming coincidence at 5461 A. They are plotted in figure 1. They represent the observed values of relative luminosity for the present group of 20 observers, who have a mean Y/B ratio 1·021.

From figure 1 it is seen that the values in line F yield a curve very similar in shape to the standard curve, but lying rather higher in the region extending from 5550 to 6300 A., while the values at 6439 A. lie somewhat below the standard curve. It may here be noted, with reference to the remarks under paragraph (ii) above, that the new curve is representative of winter conditions, except for a possible slight modification near 5780 A. which is due to the use of the mean value of the three determinations at that wave-length.

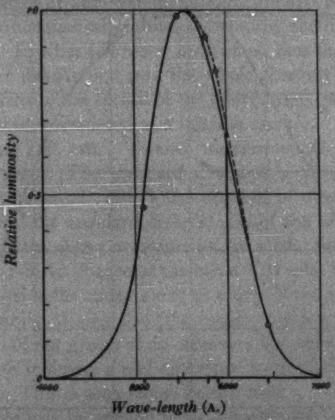

Figure 1. Curves of relative luminosity; ———, C.I.E. standard; - - - -, observed results; ⊙, mean observed values.

(iv) *Logarithmic luminosity curves.* The way in which figure 1 is plotted, though common, is not the best for the purpose of examining the departure of the observed values from the standard curve. For this reason a portion of the standard curve together with the observed points from line F of table 1 are plotted on a logarithmic scale in figure 2. It may be noted that the apparently symmetrical summit of the standard curve in figure 1 is now seen to be asymmetrical. There is of course no a priori reason for expecting symmetry, if the curve is regarded as the sum of three trichromatic sensation curves. It is apparent, however, that the present results support a more symmetrical shape, not necessarily differing appreciably from the standard curve in the regions comprising wave-lengths shorter than 5600 A. or longer than 6300 A. Between these limits the observed curve is above the standard, the difference being a maximum of 11 per cent at about 5950 A.

By the use of the logarithmic curve drawn through the observed points, a fairly accurate interpolation of intermediate values of relative luminosity can be made.

Over the range where the present determination differs from the standard values, the new values obtained are as shown in table 2.

Table 2

λ	5700	5800	5900	6000	6100	6200	6300
K	0·977	0·935	0·830	0·703	0·547	0·401	0·275

From these values the following calculations can be made.

(v) *Calculation of Y/B ratio on the basis of the observed luminosity.* Spectrophotometric data were available on the actual filters used for the Y/B ratio

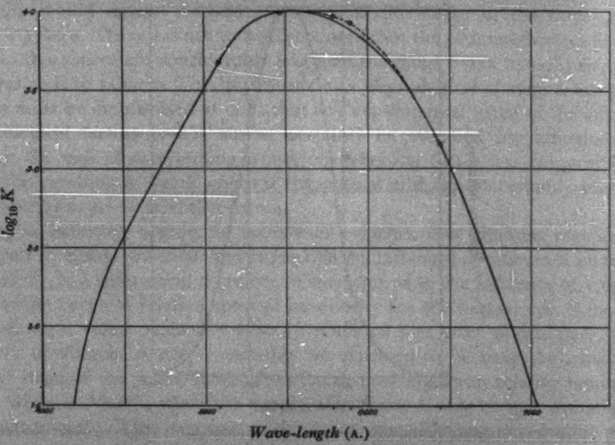

Figure 2. Logarithmic curve of relative luminosity.

measurements given in table 1. These enabled a calculated Y/B ratio to be obtained from the measured luminosity values given in paragraph (iv) above, as well as a similar figure based on the standard values. These Y/B values were:

Y/B derived from standard values of *K*, 0·979;
Y/B derived from new values of *K*, 0·996.

Thus the mean observed Y/B ratio, namely, 1·021, is some 2½ per cent higher than that calculated on the basis of the luminosity curve taken as representing the mean for the 20 observers employed. Moreover, the calculated value 0·979 based on the standard curve is also some 2 per cent below unity, which was the mean observed Y/B ratio of the large group of observers on which the standard curve was based. The fact that in both these cases the difference between the observed and calculated values of the Y/B ratio is roughly the same suggests that the standard luminosity curve does not represent the average eye in some wave-length-region outside the range (5086 to 6439 A.) of the present determination⁽¹¹⁾. In any case, the present measurements afford no explanation of the fact that the mean observed Y/B ratio for the present group of 20 observers differs by 2½ per cent from that calculated on the spectrophotometric basis from the observed mean luminosity curve. Obviously, however, measurements of relative luminosity at five wavelengths only cannot give sufficient information to elucidate this point or to determine precisely the form of the mean luminosity curve.

(vi) *Statistical considerations.* Before it can be said that the present results are, or are not, consistent with a universal mean Y/B ratio of unity for normal observers, or with the accepted standard luminosity function, or with both, it is necessary to know not only the range of any systematic experimental error, but also the uncertainty introduced on statistical grounds by limiting the observations to a group of 20 observers only. In what follows it is assumed that the quantities concerned conform to a Gaussian distribution over the whole observer population, and that a sufficiently good estimate of the spread of the distribution can be obtained from the observer group available, considered as a random sample.

Consider first the Y/B ratio. Twenty measurements are available with an arithmetic mean of 1·021. The standard deviation σ is $\sqrt{\{\Sigma\Delta^2/(N-1)\}}$, where Δ is the difference from mean, and N the number of observations; its value for the group is 0·061, and the standard error $\sigma/\sqrt{N}$ of the mean is 0·0135. Double this figure is 0·027. If the above assumptions are made, therefore, it follows from theory that there is a 1-in-20 chance of the mean Y/B ratio for a random sample of 20 observers lying outside the range 1·027 to 0·973, if the true mean value for the whole population is unity. Using this (the usual) criterion, one may say that only if the mean Y/B ratio of our group of 20 observers lies outside this range, can it be inferred that the result indicates a real difference from unity of the mean value for the whole population. This is seen not to be the case.

The same principle may be applied to the measurements of relative luminosity. It is found that at 5086 and 6439 A. there is no inconsistency between the observed values and the standard C.I.E. values. At 5780 A. the degree of inconsistency is small and is not beyond limits of possible systematic error, of a type such as that associated with the photocell calibration. At 5893 A. the result is inconsistent, as judged by the criterion chosen above, with the standard value, but is consistent with a value some 5 per cent above it. Again, any systematic error may affect this figure by roughly ± 2 per cent of the luminosity value.

The above calculations are based on each observed quantity separately. Taken as a whole, the luminosity measurements may be shown to indicate a curve differing from the standard curve in the yellow region of the spectrum to which the figures in paragraph (iv) above apply. The possible error (including systematic error) of these figures may be taken as approximately 4 or 5 per cent of the values given.

The difference may, however, be due, among other possible causes, to differences in photometric conditions. The conditions applicable to the measurements on which the standard curve is based varied to some extent⁽¹²⁾, and in many cases a larger field-size than in the present work was used. It is perhaps regrettable that recent workers in this field have not generally conformed to conditions of field-size and field-brightness now considered acceptable in heterochromatic photometry. Further, the whiteness of the photometric test surface is not specifically mentioned in some cases.

§ 7. CONCLUSIONS

(1) Measurements of relative visual luminosity of radiation at five wave-lengths by a group of 20 observers under winter conditions result in a curve differing from the C.I.E. standard luminosity curve in the region 5600 to 6300 A. The difference (excess of observed value over standard value) is a maximum, and amounts to about 11 per cent, at 5950 A. The values shown in table 3 are obtained by interpolation on a curve of logarithms of observed values, relative to the standard value of 0·984 at 5461 A.

Table 3

λ	5700	5800	5900	6000	6100	6200	6300
K	0·977	0·935	0·830	0·703	0·547	0·401	0·275

The error in these figures is estimated at ± 4 to 5 per cent of their values.

(2) The photometric conditions were as follows: Guild flicker photometer, with magnesium oxide test surface; field-diameter 2°; field-brightness 20 to 35 equivalent metre candles, with surround field; unrestricted pupil; no optical system between eye and photometer screen. It is emphasized that these conditions are not in all respects identical with those for which the standard curve applies. They are, however, now generally adopted in heterochromatic photometry.

(3) Seasonal variations appear in certain of the observations, and the suggestion is put forward that these, and possibly also the departure from the standard luminosity curve, may be due to physiological causes such as variation in vitamin supply.

(4) Measurements of the Y/B ratio of the observers give a mean value of 1·021, which is not statistically inconsistent with selection at random from a population having a mean Y/B ratio of unity.

(5) A simple statistical study of the luminosity measurements indicates the desirability of using groups of observers considerably larger than 20 in order to obtain a desirably low value of the statistical probable error.

(6) Attention is finally called to the desirability of considering, in comparisons with the standard luminosity function, only such measurements as conform closely to the appropriate photometric conditions.

§ 8. ACKNOWLEDGEMENTS

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OBLIQUE REFRACTION THROUGH A CYLINDER

By H. W. LEE, Scophony Laboratories

Received 30 *March* 1939. *Read* 23 *June* 1939

ABSTRACT. The power and the first-order aberration of a cylindrical lens are found when the incident light is not normal to the axis of the cylinder.

J. H. McLeod and F. E. Altman have stated(1) in effect that the power of a cylinder for a bundle of rays making an angle θ with the normal section is $F \sec \theta$, where F is the power of the normal section. This is incorrect, the fallacy arising from the fact that refraction takes place in a plane in the glass making an angle ϕ with the normal section, where $\sin \theta = n \sin \phi$. A little consideration will show that the correct formula for the power of an oblique section is the one familiar in the investigation of the astigmatism of spherical surfaces, $(n \cos \phi - \cos \theta)/r$, where r is the radius of the normal section. As little has been published about the optics of cylindrical surfaces, it may be of interest to derive this formula independently.

Let AP, BQ be a pair of parallel rays falling on a cylinder at an angle θ to the normal section, and meeting it at P and Q respectively, AP being in a radial plane of the cylinder while $APQB$ is at right angles to it. Let O be the centre of the normal section at P; PQ is the oblique section of the cylinder and is therefore an ellipse having semiaxes r and $r \sec \theta$. PM is the major semiaxis, and PM' is the refracted ray corresponding to PM. From Q, QN is drawn perpendicular to PM, and NO' perpendicular to the cylinder axis. Then $O'Q$ is the normal to the surface at Q. Let NQ be a, the semiaperture of the beam.

Then we have

$$PM = r \sec \theta, \quad NM = (r^2 - a^2)^{\frac{1}{2}} \sec \theta,$$

$$NP = \{r - (r^2 - a^2)^{\frac{1}{2}}\} \sec \theta$$

from the ellipse PQ. We may write

$$NM = cr \sec \theta,$$

$$NP = (1 - c)\, r \sec \theta,$$

where $rc = (r^2 - a^2)^{\frac{1}{2}}$.

Take O as origin and axes as drawn; then the co-ordinates of Q are $r\,(1 - c) \tan \theta$, a, cr, and the co-ordinates of O' are $r\,(1 - c) \tan \theta$, 0, 0, while the direction cosines of $O'Q$ are 0, a/r, c, and the direction cosines of BQ are $-\sin \theta$, 0, $\cos \theta$. Hence if i is the angle of incidence at Q,

$$\cos i = c \cos \theta,$$

and if i' is the refraction, $\qquad n \sin i' = \sin i,$

where n is the refractive index. The direction cosines of the refracted ray are thus

$$\sin\theta/n,\quad -(n\cos i'-c\cos\theta)\,a/nr,\quad -\{nc\cos i'+(a^2/r^2)\cos\theta\}/n,$$

and so the equations of the refracted ray are

$$\frac{x-r(1-c)\tan\theta}{-\sin\theta}=\frac{(y-a)\,r}{a(n\cos i'-c\cos\theta)}=\frac{(z-cr)}{nc\cos i'+(a^2/r^2)\cos\theta},$$

and the ray meets the corresponding ray which has $-a$ instead of a, and the plane $y=0$, in the point whose co-ordinates are ξ, ζ, where

$$\xi=r\sin\theta/d+r(1-c)\tan\theta,$$

$$\zeta=-r\cos\theta/d,$$

and $d=n\cos i'-c\cos\theta$.

So far a has been unrestricted. In accordance with the usual procedure in finding the focus we suppose a to be so small that its square may be neglected. Then

$$c=1,$$

$$\cos i=c\cos\theta=\cos\theta;$$

$$\therefore\ i=\theta,$$

and so

$$i'=\phi,$$

where $n\sin\phi=\sin\theta$. Here ϕ is the angle of refraction corresponding to θ and

$$d=n\cos\phi-\cos\theta.$$

The distance of $(\xi, 0, \zeta)$ from $P(0, 0, r)$ is $\{\xi^2+(\zeta-r)^2\}^{\frac{1}{2}}$ and this being in glass $=nf$, where f is the focal length for this oblique refraction;

$$\begin{aligned}\therefore\ nf&=r\{\sin^2\theta+(\cos\theta+d)^2\}^{\frac{1}{2}}/d\\&=(r/d)\{\sin^2\theta+n^2\cos^2\phi\}^{\frac{1}{2}}\\&=(r/d)\,n,\end{aligned}$$

since $\sin\theta=n\sin\phi$.

The power $1/f$ therefore

$$=\frac{d}{r}=\frac{n\cos\phi-\cos\theta}{r}=\frac{\sin(\theta-\phi)}{r\sin\phi}.$$

The power of the normal section is $(n-1)/r$, so that

$$\text{oblique power}\div\text{normal power}=\sin(\theta-\phi)/(n-1)\sin\phi.$$

The table shows the values of various quantities for different values of θ. In the fifth column are values deduced from a simple formula which gives the value of the oblique power with sufficient accuracy. The sixth column shows how the power would vary according to McLeod and Altman's statement.

The formula is capable of verification with any cylindrical lens of sufficiently long focal length. That McLeod and Altman found experimental verification for their formula can only be accounted for by the very short focal length of their lens, which must have made accurate measurements impossible.

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$$\xi=r\sin\theta/d+r(1-c)\tan\theta,$$
$$\zeta=-r\cos\theta/d,$$

and $d=n\cos i'-c\cos\theta$.

So far a has been unrestricted. In accordance with the usual procedure in finding the focus we suppose a to be so small that its square may be neglected. Then

$$c=1,$$
$$\cos i=c\cos\theta=\cos\theta;$$
$$\therefore\ i=\theta,$$

and so
$$i'=\phi,$$

where $n\sin\phi=\sin\theta$. Here ϕ is the angle of refraction corresponding to θ and

$$d=n\cos\phi-\cos\theta.$$

The distance of $(\xi, 0, \zeta)$ from $P\,(0, 0, r)$ is $\{\xi^2+(\zeta-r)^2\}^{\frac{1}{2}}$ and this being in glass $=nf$, where f is the focal length for this oblique refraction;

$$\begin{aligned}\therefore\ nf&=r\{\sin^2\theta+(\cos\theta+d)^2\}^{\frac{1}{2}}/d\\&=(r/d)\{\sin^2\theta+n^2\cos^2\phi\}^{\frac{1}{2}}\\&=(r/d)\,n,\end{aligned}$$

since $\sin\theta=n\sin\phi$.

The power $1/f$ therefore

$$=\frac{d}{r}=\frac{n\cos\phi-\cos\theta}{r}=\frac{\sin(\theta-\phi)}{r\sin\phi}.$$

The power of the normal section is $(n-1)/r$, so that

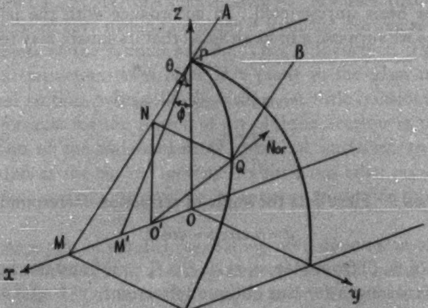
$$\text{oblique power}\div\text{normal power}=\sin(\theta-\phi)/(n-1)\sin\phi.$$

The table shows the values of various quantities for different values of θ. In the fifth column are values deduced from a simple formula which gives the value of the oblique power with sufficient accuracy. The sixth column shows how the power would vary according to McLeod and Altman's statement.

The formula is capable of verification with any cylindrical lens of sufficiently long focal length. That McLeod and Altman found experimental verification for their formula can only be accounted for by the very short focal length of their lens, which must have made accurate measurements impossible.

Table

θ (deg.)	ϕ (deg.) $n=1{\cdot}525$	$\dfrac{\sin(\theta-\phi)}{\sin\phi}$	$\dfrac{\text{Oblique power}}{\text{Normal power}}$	$1+\dfrac{(\theta \text{ deg.})^2}{10^4}$	$\sec\theta$
0	0	0·525	1	1	1
5	3·276	0·52632	1·0025	1·0025	1·0038
8	5·236	0·52836	1·0064	1·0064	1·0098
10	6·5983	0·53028	1·0100	1·0100	1·0154
12	7·8359	0·53260	1·0145	1·0144	1·0223
15	9·7714	0·53695	1·0228	1·0225	1·0353
18	11·6910	0·54231	1·0330	1·0324	1·0515



First-order aberration. The focus is at a distance nf along PM', whose direction cosines are $(\sin\phi, 0, \cos\phi)$, and so the co-ordinates of the focus are

$$nf\sin\phi,\ 0,\ r-nf\cos\phi,$$

or

$$f\sin\theta,\ 0,\ r-nf\cos\phi.$$

The oblique cylindrical aberration thus has co-ordinates given by the following equations:

$$d\xi = -(f-r/d)\sin\theta + r(1-c)\tan\theta,$$

$$d\zeta = nf\cos\phi - (r/d)\cos\theta - r.$$

Now $\quad c=(1-a^2/r^2)^{\frac{1}{2}} = 1-\frac{1}{2}a^2/r^2$ neglecting higher powers of $1/r$,

$$\cos i = c\cos\theta = (1-\tfrac{1}{2}a^2/r^2)\cos\theta;$$

$$\therefore\ \sin^2 i = 1-\cos^2\theta\,(1-a^2/r^2) = \sin^2\theta + (a^2/r^2)\cos^2\theta.$$

$$\therefore\ \sin i = \sin\theta\,(1+\tfrac{1}{2}a^2\cot^2\theta/r^2),$$

$$\sin i' = \sin\phi\,(1+\tfrac{1}{2}a^2\cot^2\theta/r^2);$$

$$\therefore\ \cos^2 i' = 1-\sin^2\phi\,(1+a^2\cot^2\theta/r^2)$$

$$= \cos^2\phi - a^2\cos^2\theta/n^2r^2.$$

So $$\cos i' = \cos\phi\,(1 - \tfrac{1}{2}a^2\cos^2\theta/n^2r^2\cos^2\phi),$$

and $$\begin{aligned} d &= n\cos i' - c\cos\theta \\ &= n\cos\phi\,(1 - \tfrac{1}{2}a^2\cos^2\theta/n^2r^2\cos^2\phi) - (1 - \tfrac{1}{2}a^2/r^2)\cos\theta \\ &= n\cos\phi - \cos\theta + \tfrac{1}{2}(a^2/r^2)\cos\theta\,(1 - \cos\theta/n\cos\phi) \\ &= (r/f)\,(1 + \tfrac{1}{2}a^2\cos\theta/nr^2\cos\phi), \end{aligned}$$

and $$r/d = f\,(1 - \tfrac{1}{2}a^2\cos\theta/nr^2\cos\phi).$$

So $$\begin{aligned} d\xi &= -\tfrac{1}{2}fa^2\cos\theta\sin\theta/nr^2\cos\phi + \tfrac{1}{2}(a^2/r)\tan\theta \\ &= \tfrac{1}{2}(a^2/r)\tan\theta\,(1 - f\cos^2\theta/nr\cos\phi), \end{aligned}$$

and $$\begin{aligned} d\zeta &= nf\cos\phi - r - f\cos\theta\,(1 - \tfrac{1}{2}a^2\cos\theta/nr^2\cos\phi) \\ &= \tfrac{1}{2}a^2f\cos^2\theta/nr^2\cos\phi. \end{aligned}$$

$d\xi$ may be written $$\begin{aligned} &\tfrac{1}{2}f(a^2/r)\tan\theta\,\{(n\cos\phi - \cos\theta)/r - \cos^2\theta/nr\cos\phi\} \\ &= \tfrac{1}{2}(a^2f/r^2)\tan\theta\,\{(n^2\cos^2\phi - 1 + n^2\sin^2\phi)/n\cos\phi - \cos\theta\} \\ &= \tfrac{1}{2}(a^2f/r^2)\sin\theta\,\{(n^2-1)/n\cos\theta\cos\phi - 1\}, \end{aligned}$$

which shows that $d\xi$ is positive, i.e. that the aberration has lifted the ray above the plane of refraction (this is also obvious from consideration of the figure).

When $\theta = 0$,

$$d\xi = 0, \quad d\zeta = \tfrac{1}{2}a^2f/nr^2,$$

the ordinary spherical aberration.

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THE THIRD SPARK SPECTRUM OF KRYPTON, Kr IV

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ABSTRACT. The spectrum of a highly condensed discharge through fine capillary tubes containing krypton is studied with the help of a large quartz Littrow spectrograph. An examination of the data led to the identification of the multiplets $5s\,^4P-5p\,^4D$, $5s\,^4P-5p\,^4P$, etc., occurring in the near ultra-violet. The important intervals $5s\,^4P_{\frac{1}{2}}-5s\,^4P_{1\frac{1}{2}}$ and $5s\,^4P_{1\frac{1}{2}}-5s\,^4P_{2\frac{1}{2}}$ are 3100 and 3223 cm.$^{-1}$ respectively. A term scheme is set up, involving the classification of about 60 lines.

§ 1. INTRODUCTION

In continuation of the analysis of Br III⁽¹⁾ carried out in this laboratory, an investigation of the spectrum of Kr IV was undertaken. Of the several spectra of krypton, the analyses of Kr I⁽²⁾, Kr II⁽³⁾, Kr III⁽⁴⁾ are known to a considerable extent through the work of Boyce, De Bruin and others. But in Kr IV only three combinational lines $4p\,^4S-sp^4\,^4P$ in the extreme ultra-violet were identified and assigned tentatively by Boyce⁽⁴⁾. These three lines, however, do not help in extending the analysis of Kr IV to the quartz region as the ^4P term that is involved arises from the sp^4 configuration. An attempt is made in the present work to identify the multiplets $5s\,^4P-5p\,^4D$, $5s\,^4P-5p\,^4P$, etc., occurring in the near ultra-violet. Our knowledge of the structure and analysis of Br III formed, to a large extent, the basis for identifying these multiplets, since doubly-ionized bromine and trebly ionized krypton are isoelectronic. A preliminary report of the results obtained has appeared in *Current Science*⁽⁵⁾.

§ 2. EXPERIMENTAL

The usual H-type discharge tube closed at one end by a quartz plate was provided with an arrangement for enclosing a certain volume of the gas. The capillary portion was about 8 in. long and ½ mm. or 1 mm. in diameter. The gas, which was reported to be about 99-per cent-pure, was contained in a quarter-litre pyrex flask provided with an internal sealed-in joint. The flask as a whole was connected to the discharge tube by a side attachment with glass stoppers between the two. By means of an external magnet and a small iron nail inside the neck of the flask, the sealed-in joint could be broken, so that a connexion could be established between the gas and the discharge tube whenever desired. The discharge tube was at first exhausted, and then a small quantity of the gas was admitted into the region between the stoppers and finally into the evacuated discharge tube. This process would protect the main reservoir of the gas from being contaminated by any gaseous impurities that might be contained in the discharge tube itself. In the actual experiment, the process of exhausting and filling the discharge tube with the gas was repeated to ensure purity of the gas whose spectrum was studied. Figure 1 represents a diagram of the discharge tube, the drying system of calcium chloride and potassium hydroxide towers, and the gas reservoir and its attachments. A condensed discharge was passed through the tube between aluminium electrodes. The excitation was by a transformer giving about 20,000 V. in the secondary. A small inductance of about 0·05 mH., used in series with the secondary circuit, suppressed the lines due to residual air in the discharge tube.

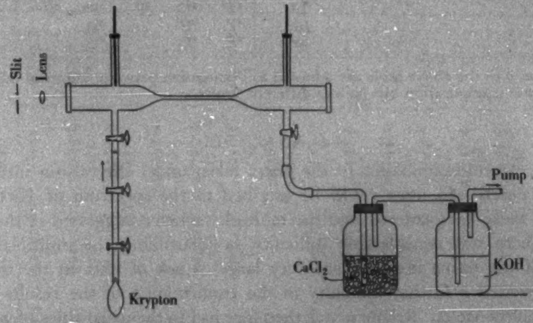

Figure 1

The usual method of varying the intensity of excitation by altering the capacity and the length of the series gap was adopted for the purpose of obtaining photographs of the spectrum in different stages of excitation. In addition the gas pressure inside the tube was varied; but when this was a little too high, the lines obtained were too broad and unfit for measurement. A pressure of 0·1 mm. gave fairly good spectra. Hilger medium and Littrow types of quartz spectrographs were employed for photographing the spectra. The wave-lengths of nearly all the lines were measured from the Littrow plates.

§ 3. PREDICTED TERMS

Table I gives the terms determined theoretically as characteristic of the spectrum of Kr IV in accordance with the theory of Hund and Heisenberg. Out of these the 5*s* and 5*p* terms were identified in the present work, along with some of the *md* terms. A complete elucidation of the structure of Br III is being carried out by

Dr K. R. Rao and this is expected to help in the further identification of the other important terms in Kr IV.

Table 1. Predicted terms in Kr IV

Configuration n_l								Term prefix	Terms						
4_0	4_1	4_2	5_0	5_1	5_2	5_3	6_0								
2	3	.	.	.	.	.	.	$4p$	^4S	.	.	^2D	.	.	^2P
2	2	.	1	.	.	.	.	$5s$	^4P	^2P	.	^2D	.	.	^2S
									^4D	^4P	^4S	^2F	$^2D^a$	.	$^2P^b$
2	2	.	.	1	.	.	.	$5p$	^2D	^2P	^2S	$^2P^a$	.	.	.
									^4F	^4D	^4P	^2G	$^2F^a$	$^2D^a$	$^2D^b$
2	2	1	.	.	.	.	.	$4d$	^2F	^2D	^2P	$^2P^a$	$^2S^a$	.	.
2	2	.	.	.	.	.	1	$6s$	^4P	^2P	.	^2D	.	.	^2S
1	4	.	.	.	.	.	.	sp^4	^4P	^2P	.	^2D	.	.	^2S

Letters *a*, *b* in the above table are adopted to distinguish between identical terms arising from the same configuration but having different limits.

§ 4. ANALYSIS

After a careful examination of the plates taken under the various intensities of discharge, nearly 180 lines could be ascribed to the spectrum of Kr IV. More than 50 of these have entered into the multiplet scheme suggested in the ensuing section. There was considerable difficulty in identifying the multiplets as the intervals of the terms involved are very large. Lack of data in the Schumann region presented a further difficulty in the confirmation of the results obtained in the near ultra-violet. Recourse was therefore had to the usual rules of comparing the screening constants and the frequencies of the corresponding lines in the other isoelectronic spectra. From table 2 the intervals $5s\,(^4P_{\frac{1}{2}}-{}^4P_{1\frac{1}{2}})$ and $5s\,(^4P_{1\frac{1}{2}}-{}^4P_{2\frac{1}{2}})$ could be predicted.

Table 2

Spectrum	$5s\,(^4P_{\frac{1}{2}}-{}^4P_{1\frac{1}{2}})$	S	ΔS	$5s\,(^4P_{1\frac{1}{2}}-{}^4P_{2\frac{1}{2}})$	S	ΔS
As I	915·8	18·94	0·81	1287·7	17·17	0·61
Se II	1483·6	18·13	0·75	1920·7	17·06	0·31
Br III	2253·5	17·38	0·45	2589·1	16·75	0·01
Kr IV	3100	16·93		3223	16·74	

Our preliminary trials with the data pertaining to Kr IV indicated the existence of two important wave-number differences of the values $\nu 3100$ and $\nu 3223$, and these were found to correspond respectively to $5s\,(^4P_{\frac{1}{2}}-{}^4P_{1\frac{1}{2}})$ and $5s\,(^4P_{1\frac{1}{2}}-{}^4P_{2\frac{1}{2}})$. From table 3 the approximate positions of the important combinational lines of the

Table 3

Spectrum	$5s\ ^4P_{\frac{1}{2}}-5p\ ^4D_{\frac{1}{2}}$		$5s\ ^4P_{\frac{1}{2}}-5p\ ^4D_{1\frac{1}{2}}$		$5s\ ^4P_{1\frac{1}{2}}-5p\ ^4D_{2\frac{1}{2}}$		$5s\ ^4P_{2\frac{1}{2}}-5p\ ^4P_{2\frac{1}{2}}$		$5s\ ^4P_{1\frac{1}{2}}-5p\ ^4D_{1\frac{1}{2}}$		$5s\ ^4P_{2\frac{1}{2}}-5p\ ^4D_{2\frac{1}{2}}$	
As I	11332	7697	11704	7737	11672	7642	11551	7573	10788	7170	10385	7009
Se II	19029	8735	19441	8981	19314	8925	19124	8938	17958	8211	17394	8256
Br III	27764	8847	28422	9718	28239	9986	28062	10248	26160	8871	25650	9351
Kr IV	36611		38140		38225		38310		35040		35001	

Table 4. Multiplet scheme of Kr IV

5*p*	$^4\mathrm{D}_{\frac{1}{2}}$	*1529* $^4\mathrm{D}_{1\frac{1}{2}}$	*3183·6* $^4\mathrm{D}_{2\frac{1}{2}}$	*3307·5* $^4\mathrm{D}_{3\frac{1}{2}}$	$^4\mathrm{P}_{\frac{1}{2}}$	*488* $^4\mathrm{P}_{1\frac{1}{2}}$	*348* $^4\mathrm{P}_{2\frac{1}{2}}$	$^4\mathrm{S}_{1\frac{1}{2}}$	A	*436* B	*2267·9* C
5*s*	36611·8	38140·9	41324·5	44632	41862	42350	42698·5	42781	44026·9	44463·4	46731·3
$^4\mathrm{P}_{\frac{1}{2}}$ = 0 *3100*	36611·8 (3)	38140·9 (7)	—	—	41862·4 (3)	42352·6 (4)	—	42781·3 (4)	—	—	—
$^4\mathrm{P}_{1\frac{1}{2}}$ = 3100 *3223*	33511·3 (3)	35040·6 (5)	38225·1 (8)	—	38763·1 (2)	39250·1 (6)	39599·9 (5)	39680 (6)	40926·2 (5)	41362·7 (4)	43630·6 (6)
$^4\mathrm{P}_{2\frac{1}{2}}$ = 6323	—	31817·6 (3)	35001·3 (5)	38310·1 (10)	—	36029·3 (6)	36376·9 (9)	36457·2 (2)	37701·9 (2)	—	40407·2 (5)
a = 1608	—	36531·6 (2)	39717·9 (4)	43024 (1)	—	—	41092·4 (1)	41172 (3)	—	—	—
b = 38	—	—	41362·7 (4)	—	—	42386·8 (3)	42737·4 (2)	—	—	—	—
c = 219	—	38359·2 (5)	41542·8 (2)	—	—	42571·4 (4)	42918·2 (3)	43000·3 (1)	44246·2 (1)	44682 (3)	—
d = 1684	—	39825·5 (2)	—	—	—	—	44382 (2)	—	—	—	—
e = 6899	—	31242 (2)	—	—	34963·4 (3)	35450·6 (1)	—	—	—	—	—
f = 11700	—	—	—	—	—	30650·9 (3)	30998·9 (6)	—	—	—	—
g = 16453·5	—	—	—	—	25410·3 (5)	25895·5 (5)	26244·1 (3)	—	—	—	—
h = 7452	—	—	—	—	—	—	35249·6 (3)	35330 (3)	36574·2 (2)	—	—
i = 2059	—	—	39265·5 (5)	42571·4 (4)	—	—	40642·4 (6)	—	—	—	—
j = 1925	—	—	—	42704·9 (2)	—	—	40775·7 (4)	—	—	—	—
k = 5382	—	—	—	39250·1 (6)	—	—	—	37400 (2)	38644·8 (3)	39080·1 (4)	—
l = 4762	—	—	—	—	—	—	—	—	39265·5 (5)	39701·8 (5)	—

multiplet $5s\ ^4P$–$5p\ ^4D$ could be known. The identification of the chief multiplets was now easy, and the analysis could thereafter be extended to include some of the *4d* terms though their assignment is not found possible at the present stage of the investigation.

Table 4 shows the multiplets so far discovered. As is characteristic of the spectra of the type under consideration, the diagonal lines of the $5s\ ^4P$–$5p\ ^4D$ multiplet are very intense and occur close to one another, as can be seen in figure 2; this grouping of the lines arises from the deviations in the intervals of the $mp\ ^4D$ term. In spectra of elements of high atomic weight such irregularities are frequently observed. Another instance in Kr IV lies in the interval $5p\ (^4D_{\frac{1}{2}}$–$^4D_{1\frac{1}{2}})$, which is rather large, unlike the corresponding intervals in other spectra of the As I type. The terms identified in the present work are given in table 5. Their absolute values could not be determined as no series of any of the terms are known. The values are obtained with reference to $5s\ ^4P_{\frac{1}{2}}$, arbitrarily adopted as zero, in accordance with the usual practice in such cases.

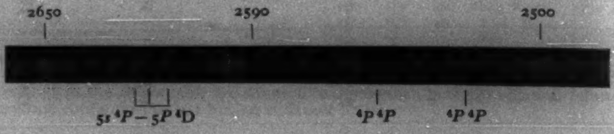

Figure 2. Discharge tube spectrum of krypton.

Table 5

Term	Term value	Δν	Term	Term value
$5s\ ^4P_{\frac{1}{2}}$	0		*a*	1608
$5s\ ^4P_{1\frac{1}{2}}$	3100	3100	*b*	38
$5s\ ^4P_{2\frac{1}{2}}$	6323	3223	*c*	219
$5p\ ^4D_{\frac{1}{2}}$	36611·8		*d*	1684
$5p\ ^4D_{1\frac{1}{2}}$	38140·9	1529·1	*e*	6899
$5p\ ^4D_{2\frac{1}{2}}$	41324·5	3183·6	*f*	11700
$5p\ ^4D_{3\frac{1}{2}}$	44632	3307·5	*g*	16453·5
			h	7452
$5p\ ^4P_{\frac{1}{2}}$	41862		*i*	2059
$5p\ ^4P_{1\frac{1}{2}}$	42350	488	*j*	1925
$5p\ ^4P_{2\frac{1}{2}}$	42698·5	348·5	*k*	5382
$5p$ A	44026·9		*l*	4762
$5p$ B	44463·4	436·5		
$5p$ C	46731·3	2267·9		

Table 6 gives the lines classified so far in the spectrum of Kr IV.

§ 5. ACKNOWLEDGEMENTS

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Table 6. Classified lines in Kr IV spectrum

λ (Int)	ν (vac)	Classification	λ (Int)	ν (vac)	Classification
3934·29 (5)	25410·3	5p ^{4}P$_{\frac{1}{2}}$–g	2546·0 (5)	39265·5	5p ^{4}D$_{2\frac{1}{2}}$–l
3860·58 (5)	25895·5	5p ^{4}P$_{1\frac{1}{2}}$–g	2524·5 (5)	39599·9	5p ^{4}P$_{2\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$
3809·30 (3)	26244·1	5p ^{4}P$_{2\frac{1}{2}}$–g	2519·38 (6)	39680·3	5p ^{4}S$_{1\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$
3261·70 (3)	30650·9	5p ^{4}P$_{1\frac{1}{2}}$–f	2518·02 (5)	39701·8	B–l
3224·99 (6)	30998·9	5p ^{4}P$_{2\frac{1}{2}}$–f	2517·0 (4)	39717·9	5p ^{4}D$_{2\frac{1}{2}}$–a
3199·91 (2)	31242	5p ^{4}D$_{1\frac{1}{2}}$–e	2510·2 (2)	39825·5	5p ^{4}D$_{1\frac{1}{2}}$–d
3142·01 (3)	31817·6	5p ^{4}D$_{1\frac{1}{2}}$–5s ^{4}P$_{2\frac{1}{2}}$	2474·06 (5)	40407·2	C–5s ^{4}P$_{2\frac{1}{2}}$
2983·22 (2)	33511·1	5p ^{4}D$_{\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$	2459·74 (6)	40642·4	5p ^{4}P$_{2\frac{1}{2}}$–i
2859·3 (3)	34963·4	5p ^{4}P$_{\frac{1}{2}}$–e	2451·7 (4)	40775·7	5p ^{4}P$_{2\frac{1}{2}}$–j
2856·2 (2)	35001·3	5p ^{4}D$_{2\frac{1}{2}}$–5s ^{4}P$_{2\frac{1}{2}}$	2442·68 (5)	40926·2	A–5s ^{4}P$_{1\frac{1}{2}}$
2853·0 (5)	35040·6	5p ^{4}D$_{1\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$	2432·8 (1)	41092·4	5p ^{4}P$_{2\frac{1}{2}}$–a
2836·08 (3)	35249·6	5p ^{4}P$_{2\frac{1}{2}}$–h	2428·04 (3)	41172	5p ^{4}S$_{1\frac{1}{2}}$–a
2829·60 (3)	35330·3	5p ^{4}S$_{1\frac{1}{2}}$–h	2416·9 (4)	41362·7	b–5p ^{4}D$_{2\frac{1}{2}}$
2774·70 (6)	36029·3	5p ^{4}P$_{1\frac{1}{2}}$–5s ^{4}P$_{2\frac{1}{2}}$	2416·9 (4)	41362·7	B–5s ^{4}P$_{1\frac{1}{2}}$
2748·18 (8)	36376·9	5p ^{4}P$_{2\frac{1}{2}}$–5s ^{4}P$_{2\frac{1}{2}}$	2406·42 (2)	41542·8	C–5p ^{4}D$_{2\frac{1}{2}}$
2742·13 (2)	36457·2	5p ^{4}S$_{1\frac{1}{2}}$–5s ^{4}P$_{2\frac{1}{2}}$	2388·05 (3)	41862·4	5p ^{4}P$_{\frac{1}{2}}$–5s ^{4}P$_{\frac{1}{2}}$
2736·65 (2)	36531·6	5p ^{4}D$_{1\frac{1}{2}}$–a	2360·4 (2)	42352·6	5p ^{4}P$_{1\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$
2733·36 (2)	36574·2	A–h	2358·5 (3)	42386·8	b–5p ^{4}P$_{2\frac{1}{2}}$
2730·55 (3)	36611·8	5p ^{4}D$_{\frac{1}{2}}$–5s ^{4}P$_{\frac{1}{2}}$	2348·27 (4)	42571·4	c–5p ^{4}P$_{\frac{1}{2}}$
2673·0 (2)	37400	5p ^{4}S$_{1\frac{1}{2}}$–K	2348·27 (4)	42571·4	5p ^{4}D$_{2\frac{1}{2}}$–i
2651·6 (2)	37701·9	A–5s ^{4}P$_{2\frac{1}{2}}$	2340·93 (2)	42704·9	5p ^{4}D$_{2\frac{1}{2}}$–j
2621·11 (7)	38140·9	5p ^{4}D$_{1\frac{1}{2}}$–5s ^{4}P$_{\frac{1}{2}}$	2339·15 (2)	42737·4	b–5p ^{4}P$_{2\frac{1}{2}}$
2615·3 (8)	38225·1	5p ^{4}D$_{2\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$	2336·75 (4)	42781·3	5p ^{4}S$_{1\frac{1}{2}}$–5s ^{4}P$_{\frac{1}{2}}$
2609·5 (10)	38310·1	5p ^{4}D$_{3\frac{1}{2}}$–5s ^{4}P$_{2\frac{1}{2}}$	2329·3 (3)	42918·2	c–5p ^{4}P$_{2\frac{1}{2}}$
2606·17 (5)	38359·2	C–5p ^{4}D$_{1\frac{1}{2}}$	2324·85 (1)	43000·3	c–5p ^{4}S$_{1\frac{1}{2}}$
2586·9 (3)	38644·8	A–K	2323·57 (1)	43024	5p ^{4}D$_{2\frac{1}{2}}$–a
2579·0 (2)	38763·1	5p ^{4}P$_{\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$	2291·26 (6)	43630·6	C–5s ^{4}P$_{1\frac{1}{2}}$
2558·08 (4)	39080·1	B–K	2259·42 (1)	44246·2	c–A
2547·0 (6)	39250·1	5p ^{4}P$_{1\frac{1}{2}}$–5s ^{4}P$_{1\frac{1}{2}}$	2252·54 (2)	44382	5p ^{4}P$_{2\frac{1}{2}}$–d
2547·0 (6)	39250·1	5p ^{4}D$_{3\frac{1}{2}}$–K	2237·34 (3)	44682	c–B
2546·0 (5)	39265·5	A–l			

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THE ELECTRICAL RESISTANCE OF FERROMAGNETIC AMALGAMS

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ABSTRACT. The electrical resistances of a series of cobalt amalgams with concentrations 0·149 to 0·548 g. of cobalt per 100 g. of mercury were measured over the temperature range 20° to 340° c. The type of {resistance, temperature} curve obtained varied with the concentration of the amalgam, but a characteristic of every amalgam on initial heating was the sharp change in resistivity, from a value less than that of pure mercury to a value considerably greater, at a temperature in the neighbourhood of 340° c. The transition temperature was found to be lowered on subsequent heating. Similar changes were observed with iron amalgams of concentrations between 0·243 and 0·497 per cent.

§ 1. INTRODUCTION

Recently J. H. Prentice⁽¹⁾ and one of us described the electrical properties of nickel amalgams with concentrations between 0·013 to 0·246 g. of nickel per 100 g. of mercury, measured over the temperature range 20° to 300° c. It was found that a very marked change in the resistance of the amalgams set in at about 225° c. and coincided with a change in the magnetic properties, the amalgams then passing from a diamagnetic to a ferromagnetic state. The change was irreversible, and it was suggested that the nickel in the freshly prepared amalgam was loosely combined with hydrogen so that the holes in the *d* bands of electrons were filled, or, alternatively, that a loose coupling obtained between the nickel and mercury atoms.

In the light of these results it was thought to be of interest to examine the electrical behaviour of cobalt and iron amalgams, although it is well known that both these ferromagnetic metals form amalgams in which the ferromagnetic constituent may be removed or concentrated by the application of a magnetic field. The magnetic properties of iron amalgams were recently investigated by Bates and Illsley⁽²⁾, and it is hoped to publish shortly an account of similar magnetic measurements which have been made on cobalt amalgams.

§ 2. EXPERIMENTAL PROCEDURE

Cobalt and iron amalgams are much less readily oxidized than nickel amalgams of similar concentration. Those used in the present work were prepared by the electrolysis of solutions of known concentrations of cobaltous sulphate or acidulated ferrous sulphate respectively with a weighed quantity of mercury as cathode, until the solutions were completely denuded of the ferromagnetic metal as shown by the disappearance of the coloured ions. The mercury used was purified by the method recommended by Bates and Baker⁽³⁾. The freshly prepared amalgams were washed in distilled water, roughly dried and transferred to a conductivity apparatus similar to that described by Bates and Day⁽⁴⁾. This apparatus had been previously degassed by heating it while it was evacuated to a temperature of about 80° c. Following the introduction of an amalgam it was evacuated and heated for a sufficient time, normally 5 to 6 hr., at a temperature between 100 and 150° c., to ensure the removal of water and of occluded gases. This procedure was most important and we made many abortive measurements with cobalt amalgams before the importance of prolonged degassing was appreciated. By frequent manipulation of the apparatus the amalgam was thoroughly mixed and a homogeneous thread was formed within the capillary tube. The apparatus was then connected through tubes respectively containing alkaline pyrogallol and drying agents to a nitrogen cylinder, and a pressure slightly above atmospheric was maintained upon the amalgam, a simple barometer tube being arranged as a safety valve.

The resistance of the amalgam thread was measured by a double potentiometer method⁽¹⁾. The temperature variation of resistance of a pure mercury thread was also measured in the same apparatus, and in the present experiments rigorous temperature control and very precise measurement of the temperature within the furnace were found to be very important. The two mercury thermometers used in earlier investigations were therefore replaced by a system of four copper constantan thermojunctions, mounted along a pyrex tube and symmetrically fixed between the two arms of the capillary tube; two thermocouples whose junctions were situated in one half of the furnace were joined in series so that independent measurement of the temperature in the two halves could be obtained. As the furnace was wound in two parts, adjustment of the furnace current in either part allowed the temperature to be made uniform throughout. The thermocouple leads were taken to a potentiometer of the type designed by Eumorfopoulos⁽⁵⁾, and the thermocouples were standardized against a standard mercury thermometer.

§ 3. RESULTS

Final measurements were made with a series of cobalt amalgams whose concentrations ranged from 0·149 to 0·548 per cent of cobalt by weight. As usual in such experiments, amalgams of higher concentration could not be used because of the difficulty of placing a homogeneous thread of proper concentration in the capillary tube. The results of these measurements are shown in figures 1 and 2 in which the values of $\Delta R/R_T$ for the several amalgams are plotted against the temperature, ΔR being the difference between the resistance of an amalgam thread and the resistance R_T of a thread of pure mercury at the temperature T in the same apparatus. In all cases the resistivity of the amalgams below 300° c. was less than that of pure mercury. The curves in figure 1 represent the results obtained when each freshly prepared cobalt amalgam was heated for the first time, whereas the curves of figure 2 represent the behaviour of the same amalgams on subsequent heating after the amalgams had cooled to room temperature.

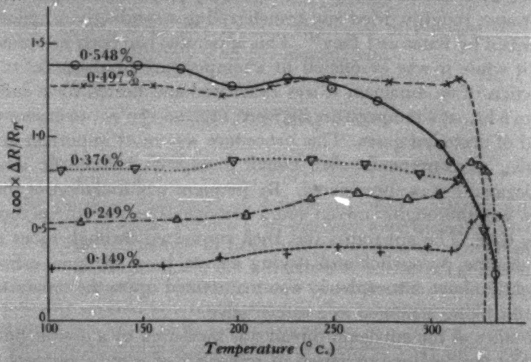

Figure 1. Variation of $\Delta R/R_T$ with temperature on initial heating for cobalt amalgams of stated percentages by weight. (The curves run practically parallel to the temperature axis from 20° to 100° c.)

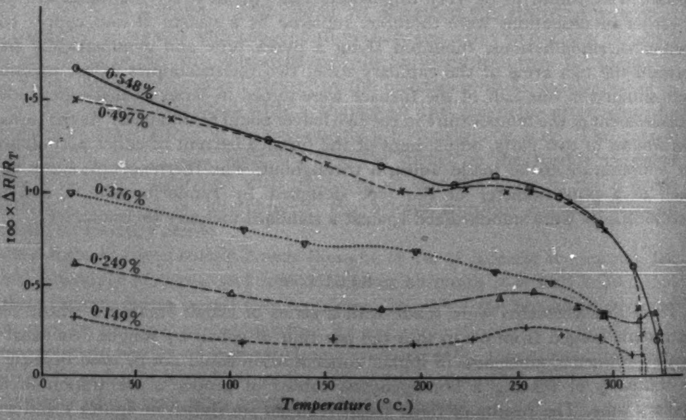

Figure 2. Variation of $\Delta R/R_T$ with temperature on subsequent heating of cobalt amalgams of stated percentages by weight.

There are some interesting features in the curves of figure 1. First we note that with all the amalgams the resistance remains practically unchanged over a wide temperature range until in the case of the weaker amalgams there is a sudden increase in ΔR, which is the less pronounced the greater the concentration of cobalt, followed by a rapid change in which ΔR changes sign. Thus at a certain temperature, which is seen to become higher as the concentration becomes lower, the resistance of the amalgam thread becomes infinite, and presumably the thread is broken, although, on cooling, electrical conduction is restored and the reproducible curves of figure 2 are obtained. The sudden increase in ΔR does not appear in the stronger amalgams. The curve for the cobalt amalgam of concentration 0·497 per cent by weight invites comment. It is to be expected that this should lie entirely between the curves for concentrations 0·548 and 0·376 per cent, and no satisfactory explanation can be offered for its departure from what was thought to be the norm. It is interpreted as an indication that, in spite of preparing and examining the amalgams by a standard technique, variations in constitution can accidentally arise.

Turning now to the curves of figure 2, it was found that the value of ΔR at room temperature was equal to the value originally obtained before heating the amalgam to a high temperature in the case of the two amalgams of concentrations 0·149 and 0·249 per cent. In the case of the remaining amalgams, there was a permanent increase in the value of ΔR at room temperature which was the greater the greater the concentration of the amalgam, and their corresponding curves in figures 1 and 2 would intersect at a temperature of about 100° C. if plotted on the same graph. It is obvious that the value of ΔR changed sign in all cases at a lower temperature than that at which the corresponding change occurred in figure 1. There seems to be no relation between the former temperature and the concentration of the amalgam, and we are inclined to think that it is of secondary importance. It is perhaps difficult to see how the sudden break in the electrical circuit can be due to any cause but the evolution of gas, presumably hydrogen, occluded in the cobalt, although it is equally difficult to explain how reproducible electrical measurements can be obtained on subsequently heating an amalgam in which such an evolution has occurred. It is still more difficult to see how the break can be due to the formation of mercury vapour, for the capillary tube was heated uniformly, a pressure greater than atmospheric was exerted upon the mercury, no trace of a break was found with mercury in the temperature range covered by these experiments, and, finally, the temperature at which the break occurred was in several cases below 320° C. Mr C. J. W. Baker in this laboratory has investigated the magnetic properties of liquid cobalt amalgams of concentrations ranging from 0·00040 to 0·0013 per cent, using an apparatus of special design which it is hoped to describe shortly in connexion with another research. Although this apparatus was not ideal for the purpose, it is at any rate the only one known to us which will give any information at all concerning the magnetic properties of liquids exhibiting ferromagnetism, and, although the concentrations which could be used were considerably less than those used in the resistance measurements, the results were of interest. They showed that the ferromagnetism at room temperature of the weakest amalgams was decreased following the initial heating, whereas that of the stronger amalgams was increased, from which it was deduced that, while some portion of the cobalt combines to form a non-ferromagnetic combination with the mercury on heating, the amalgam is quickly saturated. Moreover, the ferromagnetism approached a limiting value peculiar to each concentration at about 250° C. In these magnetic experiments, the amalgams were not heated above 330° C., but they sufficed to show that certain peculiarities in the susceptibility curves of the strongest amalgam, 0·0013 per cent, were displaced to lower temperatures on subsequent heating.

A few measurements of the resistivities of iron amalgams were also made, and curves (1) and (2) of figure 3 respectively show the results obtained on the initial and subsequent heating of an iron amalgam of concentration 0·497 per cent. The values of ΔR were always considerably less than those for cobalt amalgams of equivalent concentration. In addition, the room-temperature value of ΔR prior to heating was less than that subsequent to the heating, in agreement with the behaviour of the more concentrated cobalt amalgams. Discontinuities in the conductivity followed much the same course as in the cobalt amalgams except that humps corresponding to those in the region of 330° C., shown in figure 1, appeared to be the more accentuated the greater the concentration of iron. Indeed, with an iron amalgam of concentration 0·243 per cent we did not observe any sign of a hump, but the value of ΔR changed regularly from a small negative value at 300° C. to an equal positive value at about 320° C. and became infinite at 338° C.

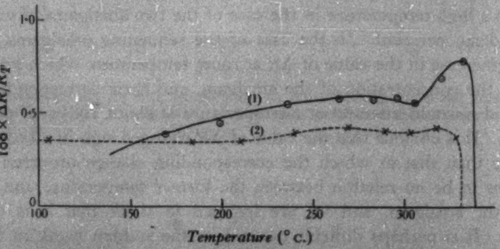

Figure 3. Variation of $\Delta R/R_T$ with temperature; for an iron amalgam of concentration 0·497 per cent by weight; (1) on initial heating; (2) on subsequent heating.

In the experiments of Bates and Prentice with nickel amalgams, the room-temperature values of ΔR were some ten times greater than those measured with cobalt amalgams, and the sudden changes observed in the neighbourhood of 225° C. caused later changes to appear insignificant in comparison, so that, unfortunately, the curves of $\Delta R/R_T$ were not followed with close attention above 300° C. We know, however, that breaks occurred at higher temperatures and that they were of a permanent character, and were undoubtedly due to the liberation of gas. Moreover, it is germane to state that, in the case of the nickel-amalgam experiments, degassing was not as rigorously pursued as in the present work. While all the nickel amalgams used by Bates and Prentice were initially diamagnetic, as were those used by Schumann⁽⁶⁾ and many other investigators, Bougault, Cattelain and Chabrier⁽⁷⁾

have recently announced the preparation of ferromagnetic nickel amalgams merely by allowing metallic nickel to be attacked by a strong acid (hydrochloric acid or dilute sulphuric acid) in the presence of mercury. We have to report that when we heated a small quantity of pure mercury, in which a small rod of pure nickel (kindly supplied by Sir Henry Wiggin and Co.) was placed, together with a small quantity of either hydrochloric acid or sulphuric acid inside an evacuated and sealed tube for 48 hr., we could not detect ferromagnetism in the mercury when it was placed in Baker's apparatus, although the surface of the nickel rod appeared amalgamated. Yet one part of ferromagnetic nickel in one million parts of mercury could readily be detected in the apparatus. If, therefore, powdered nickel was used by the French workers, we think that it is doubtful whether amalgamation was complete, and we are not convinced that nascent hydrogen is necessary for the production of nickel amalgams.

It is interesting to compare the {resistance, temperature} behaviour of these ferromagnetic amalgams with that of manganese amalgams⁽⁴⁾, for with the latter the value of $\Delta R/R_T$ showed a sudden upward trend in the region 300 to 350° C. In the present work, with the weaker ferromagnetic amalgams any upward trend is followed by the downward sweep which leads to complete breakdown. Since ferromagnetism is not an atomic phenomenon, we know that the atoms of cobalt or iron in the amalgams must be grouped together, and, from their behaviour in a magnetic field, the metals are thought to exist in a colloidal form in the amalgams. The ferromagnetic properties of iron, nickel and cobalt atoms are attributed⁽⁸⁾ to the presence of 2·5, 0·6 and 1·7 holes per atom in the d bands of electrons, respectively, while their electrical conductivities are considered due to the s electrons of which there are 0·5, 0·6 and 0·7 per atom in the s bands. It is interesting that the magnitudes of $\Delta R/R_T$ for their amalgams are in accordance with this view. Finally, on this view, a reasonable explanation of the sudden complete breaks in the {$\Delta R/R_T$, temperature} curves is surely that the mercury suddenly ceases to wet the colloidal particles of the ferromagnetic. The temperature at which this occurs would be higher for the initial heating, because the particles would then be less dispersed than on subsequent heating.

§ 4. ACKNOWLEDGEMENTS

We are indebted to the Government Grants Committee of the Royal Society for a grant with which apparatus used in these investigations was purchased.

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THE VELOCITY DISTRIBUTION IN A LIQUID-INTO-LIQUID JET. PART 2: THE PLANE JET

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ABSTRACT. The distribution of velocity in a jet issuing from a long slit has been measured and compared with two-dimensional theory. The plane jet is difficult to realize experimentally, but the experiments indicate that the distribution of velocity in the neighbourhood of the orifice approaches that given by theory if the ideal jet be considered to issue from a true line source situated behind the actual slit, which is of finite breadth, at a distance for which an expression has been found. The Reynolds' number for turbulence has been found.

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§1. INTRODUCTION

In certain investigations which I have been carrying out on the instability of a liquid-into-liquid jet, the two-dimensional case has proved of particular interest, since it is, as far as the effect of disturbances is concerned, simpler from the point of view of mathematical treatment. It therefore seemed advisable to extend the work which Mr Tsien and I(1) carried out on the circular jet to the case of a jet issuing from a long slit, as an experimental approximation to the ideal case of two-dimensional motion.

Schlichting's paper(2) gives a solution for the two-dimensional case, but it is in the form of a numerical approximation. Since then Bickley(3) has given an exact solution,* showing that if the liquid issues from a line orifice, through the origin, in the yz plane, then the x and y components of velocity are given by*

$$u = 0{\cdot}4543\,..\,(K^2/\nu x)^{\frac{1}{3}}\,\mathrm{sech}^2\,\xi, \qquad \text{......(1)}$$

$$v = 0{\cdot}5503\,..\,(K\nu/x^2)^{\frac{1}{3}}\,(2\xi\,\mathrm{sech}^2\,\xi - \tanh\xi), \qquad \text{......(2)}$$

where

$$\xi = 0{\cdot}2751\,..\,(K/\nu^2)^{\frac{1}{3}}\,yx^{-\frac{2}{3}}, \qquad \text{......(3)}$$

while M is the momentum, per unit length of the linear orifice, of fluid issuing per unit time; ρ, ν are the density and kinematic viscosity of the fluid; and $K = M/\rho$.

The object of this investigation was to see if this distribution could be realized experimentally. The general method was that described in connexion with the jet from the circular orifice: particles of aluminium were introduced into the water, both that used for the jet and that into which it discharged, and the jet was photo-

* There are two printing errors in Bickley's paper. In the expression

$$\psi = 1{\cdot}6510\,..\,(M\nu x/\rho)^{\frac{2}{3}}\tanh\xi$$

on page 729, $\frac{2}{3}$ should be $\frac{1}{3}$; and in the expression for v, at the top of page 730, γ should be ν.

graphed, with intense illumination and a fixed time of exposure. The length of the trace formed on the plate by a moving particle gives a measure of the velocity at the point in question.

§2. EXPERIMENTAL METHOD

The theory assumes that the liquid issues normally from an infinitely long orifice, of vanishingly small breadth, in an infinite plane, with a velocity the same at every point of the orifice. Practical considerations make it difficult to increase the length beyond a certain limit and at the same time to preserve a uniform velocity. The dimensions adopted for the experimental orifice were 2·03 cm. long by 0·0300 cm. wide. The cross section of the nozzle, which was made of metal, is shown in figure 1. It was provided with a plane baffle, to make the flow correspond as far as possible to the theoretical case.

A flat jet is much more difficult to maintain in a steady state than a circular jet. The slightest convection current tends to twist the plane of the jet, and slight fluctuations are very difficult to avoid even when the jet is in what must be called a stable state. For steadiness the water of the jet and the water in the tank into which it flows must be maintained at exactly the same temperature: with this in view both the reservoir and the tank were lagged, and the water was left in them both to stand overnight. Whereas with a circular jet a horizontal disposition was adopted, it was found that with a plane jet this arrangement gave rise to marked fluctuations. A vertical disposition gave much better results, and accordingly the jet was arranged so as to issue vertically downwards into a glass-sided box, as shown in figure 2, which made it easier to secure stability. A central section, normal to the plane of the jet, was isolated for observation by means of a strong beam of light from a vertical slit, focused on to the jet by a Dallmeyer 3-in. $f/1{\cdot}5$ lens. The particles were photographed with the help of a Dallmeyer 8-in. $f/2{\cdot}9$ lens.

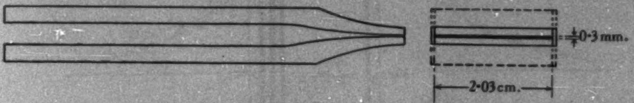


Figure 1.

The definite exposure was made by a rigid pendulum, of Helmholtz type, carrying a plate of metal with an aperture of approximately rectangular form, the two vertical edges being, however, slightly inclined so as to pass through the line of the knife-edges, which was 2 m. from the centre of the aperture. The plane of the metal plate was just in front of the slit. The pendulum was held in a displaced position by an electromagnet, so that it could be released from rest to swing through a given arc. Two arcs were actually used, giving times of exposure of 0·1353 and 0·1025 sec. respectively. These times were measured directly by focusing the slit on, and near the periphery of, a disc of sensitized paper rotated by a synchronous motor, and measuring the angle of the blackened segment. The ratio of the times was confirmed by calculation from the arcs through which the pendulum swung.

The efflux was controlled by measuring the time required for the level in the vessel *A* to fall through a measured distance, the level being measured with the Casella sensitive gauge *B* mentioned in the former paper.

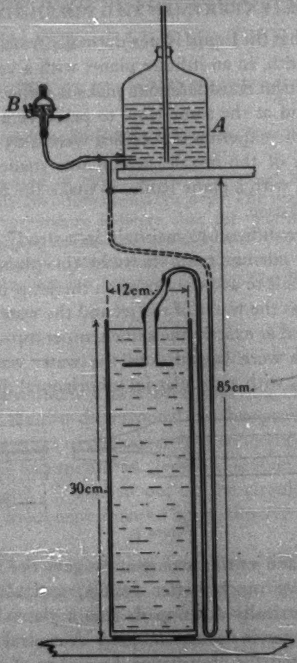


Figure 2.

§3. TREATMENT OF RESULTS

Three good plates were obtained with each of certain selected rates of outflow. The length of each trace, and the co-ordinates of its midpoint with reference to axes through the midpoint of the orifice, were measured by projecting the plate and a photographic grid, in contact with the plate, on to a horizontal screen, and reading directly. In this way velocities were obtained at a large number of points in the *x*-*y* plane. To find the velocities at selected values of *x*, correction curves were made by writing a velocity against each measured point of the *x*-*y* plane, selecting points at which the velocities were the same, and drawing curves of constant velocity. Rough curves were sufficient to give the small correction which had to be applied to convert velocities at points in the neighbourhood of the selected value of *x* into velocities at the exact value of *x* desired.

In this way experimental distributions of velocity with y at fixed values of x, 1 cm. apart, were obtained, of which examples are shown in figure 3.

It was found impossible to get the same degree of regularity in the experimental determinations as was obtained with the circular orifice. Reference has already been made to the sensitiveness of the flat jet to small disturbances; the consequence of this unsteadiness is that the experimental points show far more scatter about a regular curve than do those obtained with the circular jet. The slow jets, with R in the neighbourhood of 10, were steadier than faster ones, with R in the neighbourhood of 20 or more. At about $R=30$, definite instability with turbulence set in.

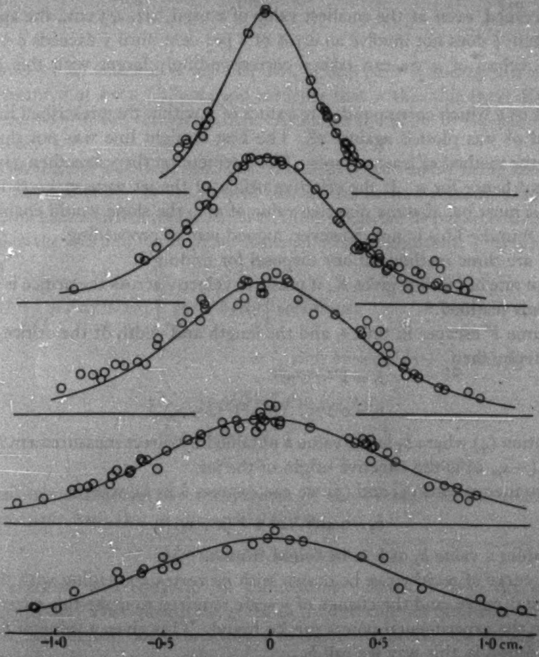


Figure 3. $R=12\cdot 8$.

Theoretically

$$u=0\cdot 454\ldots(K^2/\nu x)^{\frac{1}{3}}\operatorname{sech}^2\xi$$

$$=0\cdot 454\ldots(K^2/\nu x)^{\frac{1}{3}}\operatorname{sech}^2 by,$$

where

$$b=0\cdot 2751\ldots(K/\nu^2)^{\frac{1}{3}}\,x^{-\frac{2}{3}}, \qquad \ldots\ldots(4)$$

which, if ξ is small, becomes

$$u=a\left(1+\tfrac{1}{2}b^2y^2\right)^{-2}$$

or
$$u^{-\frac{1}{2}} = a^{-\frac{1}{2}}\,(1 + \tfrac{1}{2}b^2y^2)$$
where
$$a = 0{\cdot}454\,..\,(K^2/\nu x)^{\frac{1}{3}} \qquad \text{......(5)}$$
$$\tfrac{1}{2}b^2 = \tfrac{1}{2}\,(0{\cdot}275\,..)^2\,(K/\nu^2)^{\frac{2}{3}}\,x^{-\frac{4}{3}}$$
$$= 0{\cdot}0378\,..\,(K/\nu^2)^{\frac{2}{3}}\,x^{-\frac{4}{3}}.$$

With K equal to about 1 and ν to 0·01, which were the values with which we were concerned, even at the smallest value of x used, viz. 1·3 cm., the approximation for $\text{sech}^2\,\xi$ does not involve an error of 1 per cent until y exceeds 0·5 cm., and for larger values of x we can take y correspondingly larger with this degree of accuracy.

Values of u which corresponded to values of y within the prescribed limits were taken, and $u^{\frac{1}{2}}$ was plotted against y^2. The best straight line was put through the points by the method of least squares. The intercept on the y axis then gave a value for $a^{-\frac{1}{2}}$, and hence for a. If the effective origin of the jet were at $x=0$, instead of being, as it must be, at some negative value of x[(1)], the slope would enable a value of b to be found. This is not, however, a good way of proceeding.

There are three methods at our disposal for finding b.

(i) The rate of outflow gives K, if uniform velocity across the orifice is assumed, which seems justified.*

If volume V escapes in time t, and the length and width of the orifice are l and m respectively, then
$$K = V^2/t^2l^2m$$
and
$$b_1 = 0{\cdot}275\,..\,(K/\nu^2)^{\frac{1}{3}}\,(x+x_0)^{-\frac{2}{3}}$$
from equation (4) where b_1 is the value b obtained by direct measurement of K, and the point $(-x_0, 0)$ is the effective origin of the jet.

(ii) From equations (4) and (5) we can express b as b_2, where
$$b_2 = 0{\cdot}408\,..\,a^{\frac{1}{2}}\,\nu^{-\frac{1}{2}}\,(x+x_0)^{-\frac{1}{2}},$$
which enables a value b_2 of b to be found from a.

(iii) A curve of $\text{sech}^2\,y$ can be drawn with its vertex coinciding with that of the experimental curve, and the change of y scale required to make this curve give the best fit for the experimental points can be found. This gives a value of b directly: the value found in this way we call b_3.

A full set of readings were taken with K equal to 0·55, which means $R = um/\nu = 12{\cdot}8$, where m is the width of the orifice. The velocity distribution curves are shown in figure 3. By trial and error the best value of x_0 was found to make b_1 equal to b_2 near the jet; the following table shows the values of b_1, b_2 and b_3 with x_0 equal to 0·25. The accuracy of the determination of x_0 is not high, on account of experimental difficulties; an error of 10 per cent in x_0 is contemplated, but for present purposes a rough value suffices. As m is only 0·3 mm., an error of 0·01 mm. in the measurement of the width of the slit produces an error of 1 per cent in b_1, which involves $m^{-\frac{1}{3}}$.

* See Andrade and Tsien[(1)], p. 385.

Table 1. ($R=12\cdot8$.)

x	b_1	b_2	b_3	$0\cdot00481\times$ $b_2^4\,(x+x_0)^2/b_3$
1·3	3·62	3·62	3·60	0·550
2·3	2·59	2·53	2·30	0·558
3·3	2·07	2·04	1·83	0·572
4·3	1·76	1·63	1·30	0·540
5·3	1·55	1·39	1·20	0·463

The choice of x_0, which is at our disposal, affects both b_1 and b_2, but not b_3, so that the agreement at $x=1\cdot3$ offers good evidence that within this short distance of the orifice the flow in the central section, normal to the plane of the jet, conforms with the theory.

At larger values of x both b_2 and b_3 are less than b_1, the defect being much more marked in the case of b_3. If the flow in the central section remains plane, i.e. if strict two-dimensional streaming holds, then the momentum M per second across any line normal to the axis and to the plane of the jet must be constant. This momentum is given by

$$M=\int_0^\infty \rho u^2\,dy=2a^2\rho\int_0^\infty \operatorname{sech}^4 by\,dy \quad \int \operatorname{sech}^4 by\,dy$$

$$=\frac{1}{3b}\sinh by\operatorname{sech}^3 by+\frac{2}{3b}\tanh by$$

$$\therefore\; M=\tfrac{4}{3}\rho\, a^2/b=\tfrac{4}{3}a^2/b \text{ for water.}$$

Now
$$b_2=0\cdot408\,..\,a^{\frac{1}{2}}\,\nu^{-\frac{1}{2}}\,(x+x_0)^{-\frac{1}{2}}$$
$$=4\cdot08\,..\,a^{\frac{1}{2}}\,(x+x_0)^{-\frac{1}{2}}\quad\text{with } \nu=0\cdot01,$$

and b_3 measures b directly, by comparison of the breadth of the experimental distribution curve with that of the $\operatorname{sech}^2 y$ curve

$$\therefore\; M=0\cdot00481\,..\,b_2^4\,(x+x_0)^2/b_3$$

should be constant and equal to the momentum calculated from the measured outflow in cm.²/sec. This quantity is given in the last column of table 1, and it will be seen that, considering that b_2 is involved to the fourth power, the agreement with the outflow value $K=0\cdot55$ is good up to 4 cm. or so. The low value of b_3 means a broadening of the jet in the xy plane as compared with the theoretical value, which, if the momentum is to be maintained constant, must be accompanied by a fall in the maximum velocity a below the theoretical value, and hence a fall in b_2. As, however, the momentum involves b_2^4/b_3 the fall in b_2 is comparatively very small.

These results indicate that in the neighbourhood of the orifice the theory holds quite well; that for $1\cdot3<x<4\cdot3$ the experimental jet behaves two-dimensionally in that there is no communication of momentum, normal to the xy plane, between adjacent layers, but that it spreads somewhat more than the theory indicates: while for $x>4\cdot3$ the momentum falls off, owing presumably to a loss of momentum normal to the xy plane.

The value of x_0 is 0·25 mm., which means that

$$\frac{x_0}{m}=\frac{0{\cdot}25}{0{\cdot}03}\,\frac{R}{1{\cdot}28}=0{\cdot}65R.$$

The results at higher rates of flow did not agree very well with the two-dimensional theory. Taking $x_0=0{\cdot}65\ mR=0{\cdot}38$ when $R=19{\cdot}6$, for which a set of experiments were done, we have the following values for b_1, b_2, b_3.

Table 2. ($R=19{\cdot}6$.)

x	b_1	b_2	b_3	$0{\cdot}00481\times b_2^4(x+x_0)^2/b_3$
2·3	3·31	3·38	2·7	1·67
3·3	2·69	2·79	2·0	1·97
4·3	2·29	2·37	1·5	2·21
5·3	2·01	2·09	1·3	2·27
6·3	1·81	1·89	1·2	2·27

The value of K deduced from the outflow in cm.^{2}/sec. per cm. is 1·29.

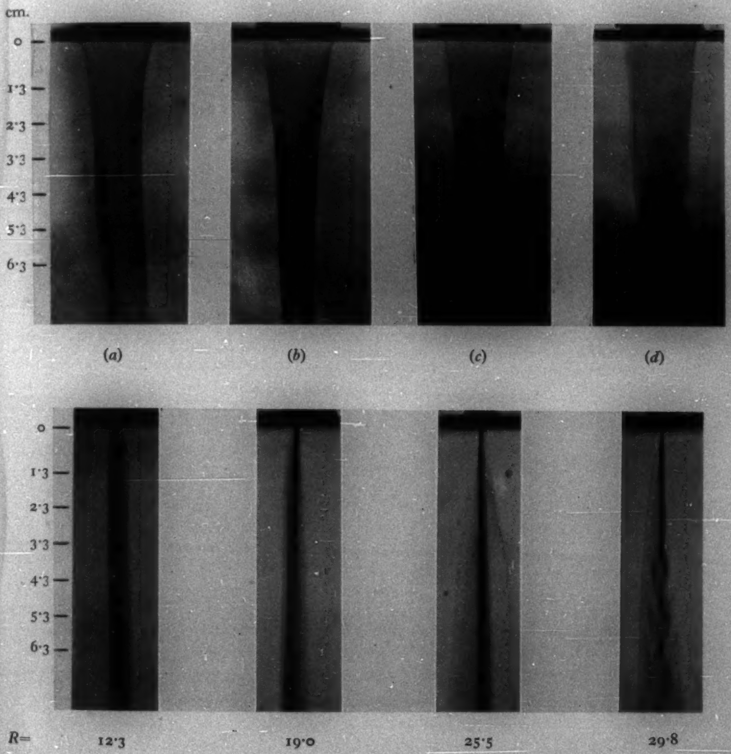
These figures indicate that there is a lateral entry of momentum into the central section which increases from $x=2{\cdot}3$ to $x=4{\cdot}3$, after which the momentum remains constant and two-dimensional conditions prevail in the central section. There are indications, however, that very near the jet the theoretical conditions hold. In particular, if we extrapolate, by considering the *b*s as a function of $(x+x_0)^{-\frac{1}{3}}$, we obtain for the value at the orifice $b_2=12{\cdot}1$, as against the value $b_1=12{\cdot}4$, which is excellent agreement. The extrapolated value of b_3 is 13·1, which is quite good, considering that it is not possible to obtain values of b_3 with high precision.

A series of measurements were also made at $R=29{\cdot}2$, which is approaching the region of instability. Here, even at 2·3 cm. from the jet, the axial velocity was considerably too high, and the spreading considerable. There is a marked lateral entry of momentum between the orifice and $x=4$, but after that the momentum in the central section is about constant with a value double that contemplated in two-dimensional theory.

Table 3. ($R=29{\cdot}4$.)

x	b_1	b_2	b_3	$0{\cdot}00481\times b_2^4(x+x_0)^2/b_3$
2·3	4·19	4·73	3·8	5·23
3·3	3·42	3·82	2·8	5·50
4·3	2·95	3·22	2·1	5·86
5·3	2·52	2·80	1·8	5·68
6·3	2·34	2·59	1·7	5·99

From these results it would appear that the ribbon jet cannot maintain its full width, especially at higher velocities. To throw light on this point observations were made with water coloured with Brilliant Green escaping into clear water. Jets with different rates of outflow were photographed in, and normal to, the plane of the ribbon, and typical results are shown in the plate. It will be seen that in all cases there is a marked shrinkage in the plane of the jet, which begins sharply at the



Above, jet seen from normal to plane of jet; below, jet seen from a point in the plane.

orifice, while at some distance from the orifice the breadth becomes constant. It is remarkable that at rates of flow in the neighbourhood of instability the shrinkage is less marked, as can be seen from (*c*) and (*d*) in the plate, the latter representing the instability which is a preliminary to turbulent flow. This instability sets in at about $R=30$, and definite turbulence occurs at about $R=33$.

In the neighbourhood of $R=13$, in spite of the shrinkage in the plane of the ribbon, the central section is unaffected at $x=1\cdot3$ cm. while, as we go further from the orifice, the central section of the jet spreads, normally to the plane of the ribbon, more than two-dimensional theory indicates, but without sensible entry of momentum. In the neighbourhood of $R=19$ the shrinkage entails both abnormal spreading and lateral entry of momentum, but there are strong indications that the theory holds in the immediate neighbourhood of the orifice. At $R=29$, just before turbulence, the lateral entry of momentum into the central section is very marked, in spite of the apparently small shrinkage of the ribbon.

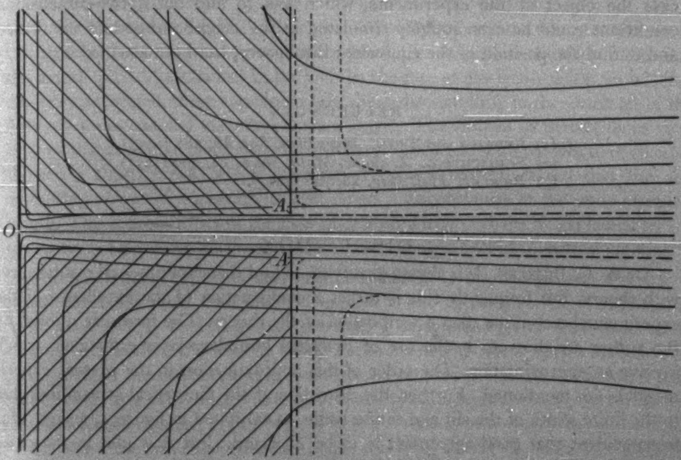


Figure 4.

We have seen that the distance x_0 of the effective origin of the jet behind the plane of the orifice is given by $x_0/m=0\cdot65R$, where m is the width of the orifice. Figure 4 shows the theoretical stream lines, given by $\psi=1\cdot651\ (K\nu x)^{\frac{1}{3}}\tanh\xi$, at equal intervals of ψ, for a source at O, with the orifice of finite width AA at a distance given by this expression. The streamlines through the two edges of the orifice are shown additionally, as broken lines, and the modification of the streamlines in the neighbourhood of the plane baffle, necessary to make them confirm to the boundary condition, has been indicated in broken lines for the streamlines near the central flow.

§4. CONCLUSIONS

At small values of R, two-dimensional theory gives tolerably good results in the neighbourhood of the finite orifice, as far as the distribution in a central plane, normal to the plane of the jet, is concerned. The origin of the theoretical jet must be taken behind the finite orifice, at a distance x_0 for which an expression is given. It may be asked why, in view of the shrinkage of the ribbon in its own plane, experiments were not made with a greater length of orifice. The answer is that if the length of the slit is made too great it becomes impossible to maintain the jet plane, the slightest disturbance causing the edges to fluctuate markedly. In any case the object of the experiments, which was to find out if two-dimensional conditions could be even roughly simulated in the neighbourhood of the orifice, and to find the position of the equivalent line source, has been attained.

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DISCUSSION

Dr W. G. Bickley. It is always gratifying to find that experimental results agree with theory, but frequently this is rather a confirmation of the excellence of the experimental technique than a verification of the theory. The formulae quoted by the author are derived by the use of Prandtl's boundary-layer equations, which involve an approximation. The order of this approximation in the present experiments is not mentioned. Further, the distortion of the theoretical streamlines due to the finite width of the slit and to the baffle, as exhibited in figure 4, makes it far from evident that good agreement is to be expected. But it is just in the close neighbourhood of the theoretical infinitesimally narrow slit that the approximation is worst, and this region is not present in the experiments. I should like to ask the author whether the predicted relation between maximum velocity and distance from the virtual origin has been verified.

The explanation of the lateral contraction of the faster jets is not obvious, but the phenomenon may be somewhat similar to the rolling up of the vortex sheet behind an aerofoil.

Author's reply. The predicted relation between maximum velocity and distance from the virtual orifice held to about the same extent as did the other theoretical relations. For $R=12\cdot8$, the following is the comparison between theoretical maximum velocity, viz. $0\cdot454\,\{\kappa^2/\nu\,(x+x_0)\}^{\frac{1}{3}}$, and measured maximum velocity.

x	1·3	2·3	3·3	4·3	5·3
Calculated $u_{\max}$	1·23	1·01	0·93	0·86	0·79
Observed $u_{\max}$	1·22	0·96	0·88	0·71	0·64

The calculated and observed values agree well near the orifice, but as we go further out the actual velocity drops below the theoretical, owing to the lateral spread which also manifests itself in the low value of b_3.

Owing to the experimental difficulties to which I have referred, the present experiments are not so well adapted to check the validity of the boundary-layer equations as are those on the circular jet carried out by Mr Tsien and myself. In these latter experiments the observed values were very consistent, and the distribution-measurement, even within about 8 mm. (about 9 orifice diameters) of the orifice, gave a very close approximation to the theory. I have pointed out that at points extremely close to the orifice the virtual source cannot give an exact representation of the flow.

L. T. MINCHIN. Can the author give any explanation of the lateral shrinkage of the plane jet shown in the upper photographs? Can he explain the effect of varying the length of parallel channel in the jet? In the case of gas-into-gas jets, in which I am most interested, there appears to be a sharp discontinuity in the character of the flow when the channel exceeds a certain length. The Bureau of Standards in 1921 showed that the curve connecting the discharge coefficient with the length of channel showed a sharp break at 0·6 diameter. We have recently found that the air-entraining power of a gaseous jet also shows a discontinuity at the same channel-length. This discontinuity in the flow phenomena may perhaps be explained by the prevention of vena contracta when the channel-length exceeds a critical amount.

AUTHOR'S REPLY. I think that it is not difficult to see in a general way why a lateral shrinkage must take place. Consider a line or tube vortex of rectangular form, such as may be considered to bound a finite plane jet of rectangular cross section. At a corner, the velocity inside due to the neighbouring vortex filament will greatly exceed that outside, and in consequence the pressure inside will be lower than that outside, and the corner will tend to round off. In general an elliptic vortex will tend to become circular, as can be seen from the well-known vibrations of an elliptic vortex about the circular form, with major axis lying alternately along one diameter and along a normal diameter. The tendency of a rectangular vortex to become first roughly elliptical and then circular explains the lateral contraction, I think, in a general way.

As regards the length of the parallel channel, Hamel (quoted by Andrade and Tsien⁽¹⁾) has shown that, for two-dimensional streaming, the flow is uniform practically right across the channel when the walls converge slightly, i.e. the velocity profile is square-headed. We showed that the same held for the circular jet. When, however, the approach to the orifice is a tube which is of uniform diameter over a sufficient length, the velocity profile takes the well-known parabolic form. With varying length of parallel channel, all velocity profiles from the square-headed (with a short uniform tube) to the parabolic (with a long uniform tube) can be obtained. I do not know if the phenomenon referred to by Mr Minchin can be explained on these lines. We have never found a vena contracta with our type of jet, nor is it to be anticipated on theoretical grounds with a gradually tapering approach to a cylindrical orifice.

THE INFLUENCE OF THE ANGLE OF INCIDENCE OF LIGHT ON THE DIFFRACTION OF LIGHT BY SUPERSONIC WAVES

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ABSTRACT. Under appropriate conditions, a central minimum appears in the curve relating the total intensity of the light diffracted by supersonic waves to the angle α between the incident light beam and the sound wave-fronts; it can be made to disappear gradually by decreasing the sound-frequency, the sound-excitation, or the length of the light-path within disturbed medium. Diagrams are given showing these facts, the decomposition of a curve of the stated type into two halves belonging to the right and left orders respectively, the behaviour of individual orders, and the influence of the length of light-path within the disturbed medium on the total intensity of the diffracted light when the other variables, including α, are kept constant, when a liquid is used as sound carrier.

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§ 1. INTRODUCTION

KORFF(1) has measured photometrically the dependence, on the angle α between incident light and the supersonic wave-fronts, of the relative intensity of the light diffracted by supersonic waves into the first-order spectra using air as the sound carrier, a sound-frequency N of about 4 Mc./sec., a length of light-path within the disturbed medium of from 1 to 3 cm., and a weak excitation such that no diffraction spectra of an order greater than unity appeared. The character of his curves is substantially that postulated by theory(1, 2). Korff shows that in liquids, owing to the relatively greater sound wave-lengths, a similarly marked dependence of the diffraction on α would only appear at frequencies N of about 50 Mc./sec. if the same lengths of light-path within the disturbed medium were used. Evidently an alternative way of producing such a dependence on α in liquids consists in dropping the restriction, imposed by Korff on his investigations, to very weak sound fields. Debye and Sears(3) had already found in their fundamental experiments, carried out with a liquid sound-carrier and a frequency of about 6 Mc./sec., an alternating rise and fall to zero in the brightness of each of a plurality of diffraction spectra as α was varied unidirectionally. The early Raman-Nath theory relating to oblique light incidence* adequately explains this behaviour of the diffraction spectra. More recently, Levi(5) has offered another experimental confirmation of the fact that this early theory holds good under suitable circumstances. On the other hand, this early theory, being entirely based on the conception of corrugations produced in the wave-fronts of the light by changes in phase, and assuming rectilinear propagation of light in the disturbed medium, leads to the conclusion that the diffraction pattern would be most prominent when $\alpha=0$, though, according to Debye(3)* the maximum intensity of each spectrum is attained at a value of α slightly different from 0. There is by no means an inherent discrepancy between these two results, as the composite brightness of all orders may well be, and in many cases is, at a maximum at $\alpha=0$, though none of the individual orders has its maximum brightness at that angle of light incidence, but it has in fact been repeatedly established in the meantime (it may, for instance, be deduced from Korff's work(1)) that under appropriate conditions the maximum total diffraction effect is obtained when α differs from 0.

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* Reference 4, especially § 4, p. 419.

Concentrating on the case of liquids as sound-carriers, we may note that Becker(6), operating with weak excitations giving first-order spectra only, states that (*a*) when $N=6$ Mc./sec. the total intensity of diffracted light had its maximum at $\alpha=0$, whereas (*b*) when $N=15$ Mc./sec. it was hardly possible to obtain both first orders at the same time—which obviously means that when $\alpha=0$ the total intensity of diffracted light was approximately zero—an inclination of the light-beam by a certain angle being required to enable the observer to see "the other first order".

The phenomenon (*b*) had already received a closer investigation and theoretical explanation from Rytow(7);† Nomoto(8) described it in the same terms as Becker. Both Rytow and Nomoto used weak excitations (first order spectra only); the frequency which they used was 30 Mc./sec., i.e. twice the frequency used by Becker. With reference to experiments carried out with N equal to about 6 Mc./sec., Nomoto remarks that the maximum of the total number of diffraction spectra is not obtained at normal incidence ($\alpha=0$) "as was, as a rule, believed up to then",‡ but that two maxima were obtained at somewhat oblique light-incidences. This again does not in itself necessarily imply the presence of two corresponding maxima in the total intensity of the diffracted light. Parthasarathy(9) investigated the effect of variations in α with excitation giving rise to the appearance of fifth-order spectra when $\alpha=0$, his frequency N being about 7 Mc./sec. The length of light-path within disturbed medium which he used may be taken from the quartz dimensions he gives as amounting to about 20 mm. Calculating total intensities of diffracted light from the visual estimates given in his table 1, one finds a very appreciably higher value when $\alpha=22'$ than when $\alpha=0$.

The effect of either reducing the acoustic wave-length or increasing the excitation from those values at which the influence of α is only weak, may accordingly be described as follows: The dependence will gradually become more marked. It will at first follow the early Raman-Nath theory(4) in that the maximum total intensity will be obtained when $\alpha=0$. On carrying the reduction of wave-length or increase in excitation further, one will observe the appearance, and gradual sharpening of two chief maxima with $\alpha=\pm\alpha_1\neq 0$. (Minor side maxima will also appear.) Where the effect is obtained with a diffraction pattern in the formation of which the spectra of orders greater than 1, if any are present, play a very subordinated part only, α_1 is the angle for which the first order attains its maximum intensity; this angle has been

* See reference (4), § 4. † See especially his figure 7. ‡ Cf. reference (4).

found theoretically and experimentally by several investigators to be the first-order Bragg angle which may be derived on the assumption of a selective reflection of the light at the supersonic wave-fronts.

For opinions as to whether the effect is preferably described as wholly a result of the particular properties of light-propagation or transmission, or as partly, and under certain conditions substantially, a result of reflection—a question which is certainly one of convenience only (see the paper by Rytow(10), including Brillouin's foreword) Parthasarathy(11), Nath(12), Nomoto(8), and Extermann(13, 14).*

§ 2. EXPERIMENTAL ARRANGEMENTS

Using, similarly to Korff(1), lengths of light-path within the disturbed medium up to about 3 cm., the author has measured photoelectrically the dependence on α of the total intensity of the portion of the non-monochromatized light from a filament lamp, diffracted by supersonic waves having N approximately equal to 10 Mc./sec. in diluted ethyl alcohol, the excitation of the sound source being such as to give rise to the appearance of fourth-order spectra when $\alpha=0$.

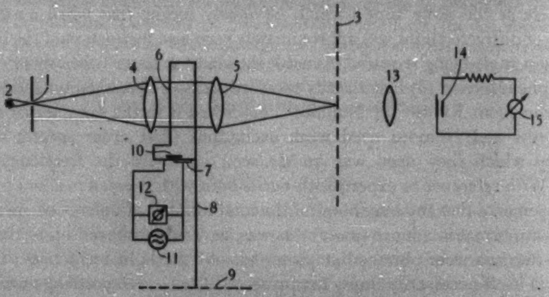

Figure 1. Experimental arrangement.

The experimental arrangement is shown in figure 1. A vertical slot 1 illuminated from a lamp 2 (of the standard filament exciter type used for sound-film purposes, having its filament disposed horizontally) served as the effective light-source, an image of which was produced in the diaphragm plane 3 by means of two lenses 4 and 5 of which the first collimated the light-beam. The supersonic wave cell 6 with the sound-producing piezo electric quartz crystal 7 was interposed in the parallel part of the light-beam with its length dimension disposed horizontally. It was capable of rotation about a vertical axis passing through its centre and had secured to one of its side walls a steel rod pointer 8 with its point arranged to move over a scale 9, divided into millimetres, as 6 was rotated. The pointer 8 could slide longitudinally so that its point could be made substantially to touch the scale 9 in any position of 6, to facilitate the reading of α. In view of the smallness of the angles involved, the largest deviations of α from zero being about $\pm 3°$, the distance, of about 20 mm., between the pointer 8 and the axis of rotation could be safely disregarded in relation to the distance, 360 mm., between the scale 9 and the axis of rotation, in other words, the readings on the scale 9 could be regarded as giving values proportional to tan α, and hence, for the small angles involved, proportional to α.

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* Note, however, that, as has been shown theoretically (see Extermann(14), especially figure 10 of his paper and appertaining discussion), the dependence of the angular effect on the acoustic wave-length is complicated and may lead, after an initial increase, to a subsequent drop in the Bragg reflection effect as the wave-length is reduced (so that N is increased).

A movable shutter 10 disposed in the cell 6 in a plane closely in front of the quartz 7 served to vary the part l, effective in setting up the supersonic wave beam, of the main horizontal dimension of the quartz. l was given directly in millimetres by the setting of the adjusting screw of the shutter 10, as this had a pitch of 1 mm.

The quartz 7 was excited from a high-frequency oscillator 11. The frequencies N were determined by means of an absorption wave-meter. For practical reasons, a valve voltmeter 12 measuring peak voltages V_p across the quartz crystal was used for determining the degree of excitation. This method affords only a rough qualitative indication of the excitation for, owing to several causes such as changes in phase, in the form of the voltage curve, and in the impedance of the quartz circuit which accompany changes in N, no simple relationship can be established between V_p and the sound-intensity.*

When the quartz crystal was excited, the image of the slot 1 in the plane 3 would spread out into the well-known diffraction pattern. A desired portion of this pattern, selected by means of a suitably shaped diaphragm introduced at 3, was projected, by means of a lens 13, on to the light-sensitive surface of a rectifier photoelectric cell 14. A galvanometer 15 in series with the photocell underwent deflections in proportion to the intensities of the selected portion of light. The diaphragm used at 3 was in most cases a simple bar shutting out the zero-order spectrum and no other spectrum, so that the galvanometer 15 measured the total diffracted light intensity. For figure 4, curves *b* and *c*, and figure 5, a large diaphragm was used having an opening whose straight vertical side edges could be displaced horizontally by means of screws.

§ 3. RESULTS

In figure 2, intensities of total diffracted light are plotted against the angle α on an arbitrary scale. Curve *a* was taken with the effective length l of the quartz crystal equal to 28 mm., and curve *b* with l equal to 16 mm. It is seen that when $l=28$ mm. the total diffracted intensity has two marked maxima with a dip between them, to which obviously the inclination where $\alpha=0$ must be ascribed, and that this effect has disappeared when $l=16$ mm., in which case the symmetrical position of the zero value of α gives maximum diffraction. This disappearance of the central dip is in qualitative agreement with the results of Korff(1).† For the theory of the influence of l see, for instances, the papers by Korff(1) and David(2); by Rytow(7), whose diagrams 7*a*, 7*b* and 7*c* show the tendency of the dip in the total intensity

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* See the footnote on page 798.

† See Korff's figure 13.

(the sum of the ordinates) to disappear as l decreases; a further by Rytow⁽¹⁰⁾, and one by Extermann⁽¹⁴⁾. In order to follow up in more detail the gradual disappearance of the dip, measurement was made of the dependence of the ratio q between the photoelectric currents produced in the maximum positions (averaged from two maxima when such were present) and central minimum position, on l at constant acoustic frequency N and constant excitation, figure 3 *a*, and on N and the excitation when l had a constant value of 28 mm., figure 3 *b*.* The disappearance of the dip is indicated by q becoming equal to unity. Figure 4 shows another curve *a* of the same character as that shown in figure 2 *a*, taken under similar conditions, and its decomposition into a left-hand and right-hand part *b* and *c*, obtained by shuttering off, in addition to the spectrum of zero order, the right-hand and left-hand spectra respectively of the diffraction pattern. Curve *d* has been formed from the sums of the ordinates of curves *b* and *c*, figure 5 shows the corresponding curves belonging to the individual first and second orders.

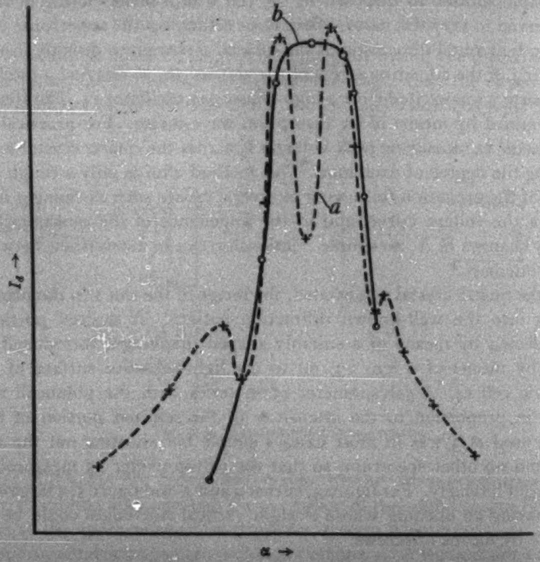

Figure 2. Dependence of the intensity I_d of the diffracted light on α for two selected values of l.

But for the fact that they have been taken with non-monochromatic light, curves 5 *a* and 5 *b* thus correspond qualitatively to the curves of Korff's figure 11⁽¹⁾,

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* It must be borne in mind that, as has been stated above, no simple relationship can be established between V_p and the sound-intensity; the four curves of figure 3 *b*, though each taken at a constant value of V_p, cannot be considered as each taken at constant excitation, since N was varied in taking them.

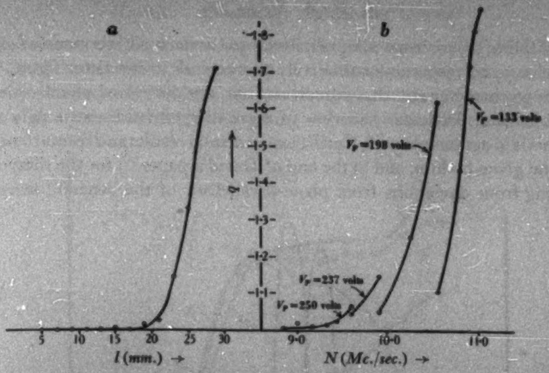

Figure 3. Curves showing the gradual disappearance of the central minimum in curves of the type shown in figure 2 on reducing *l* (curve *a*) or *N* and/or V_p (curve *b*).

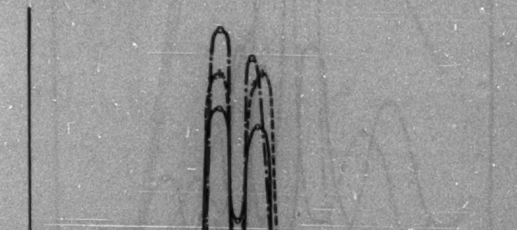

Figure 4. Dependence on α of total intensity (*a* and *d*) of diffracted light, and of the two halves of the diffraction pattern left and right of the zero order (*b* and *c*).

and one of them (5*a*) corresponds qualitatively to curve 1*a* in Extermann's figure 10(14), whilst curve 5*c* corresponds qualitatively to curve 1*b* in the latter figure. All four curves show characteristic dissymmetries, but the curves of equal order on the right and left are, particularly in view of these dissymmetries, strikingly similar in shape. This is in agreement with Korff's experimental results and seems to support the explanation given by him, and at the end of David's paper(2), for the dissymmetries, as resulting from deviations from plane-parallelism of the acoustic wave-fronts.*

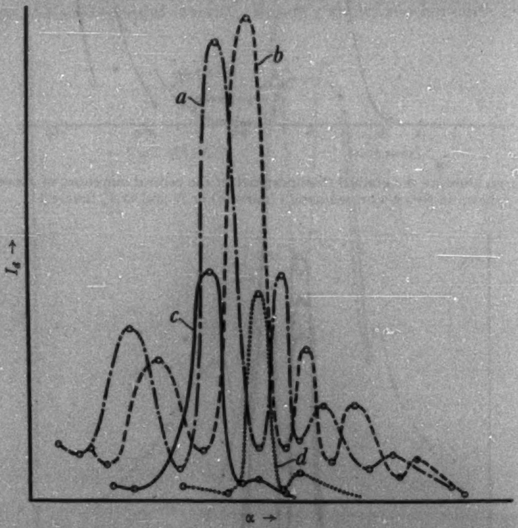

Figure 5. Dependence on α of the individual intensities of spectra of different orders of diffraction.

Considering that the light used was not monochromatic and that a photocell giving a spectral response, broader than that of the human eye, was used, the maxima and minima of the curves are remarkably pronounced. Possibly the comparatively greater sharpness of the second-order maxima predicted by Rytow(10)† would have been found if monochromatic light had been used. The phenomenon, further predicted by Rytow,† that the relative intensities of the minor maxima are smaller for the second than for the first orders, has been found in the experiments; see figure 5.

[*] Non-uniformities in the sound-ray field could be clearly observed by means of the method developed for such field observations by Bär(16), Hiedemann and Hoesch(17) and Parthasarathy(18). While the image of the supersonic cell formed by diffracted light appeared to be evenly illuminated in the positions giving the two chief maxima, adjustment to the $\alpha = 0$ position did not result in a uniform decrease in brightness of the image but in the appearance of a dark band extending in the direction of wave-propagation, showing the well-known bundle-of-rays-shaped fine structure of sound-ray fields, and flanked by parts of substantially unchanged brightness. The darkest part of this band could be made to wander from one end to the other of the band by varying α unidirectionally.

[†] See also reference 14, figure 10.

Figure 6 shows the dependence on l of the total intensity of diffracted light when α, N and the excitation are constant. Curves a, b, c were taken under conditions which, with l equal to 28 mm., resulted in the formation of a strong dip. With this value of l, α was fixed at a value giving either of the chief maxima (curves a and c) or the central minimum (curve b). Curve d was taken with N and V_p so changed as to make the central minimum disappear, α being fixed to the value giving the central maximum. Korff's$^{(1)}$ figure 14 shows results of an investigation similar to that which led to curve 6b. While none of the curves in figure 6 shows a periodic rise and fall (see Korff's paper$^{(1)}$, and particularly the detailed theory given by

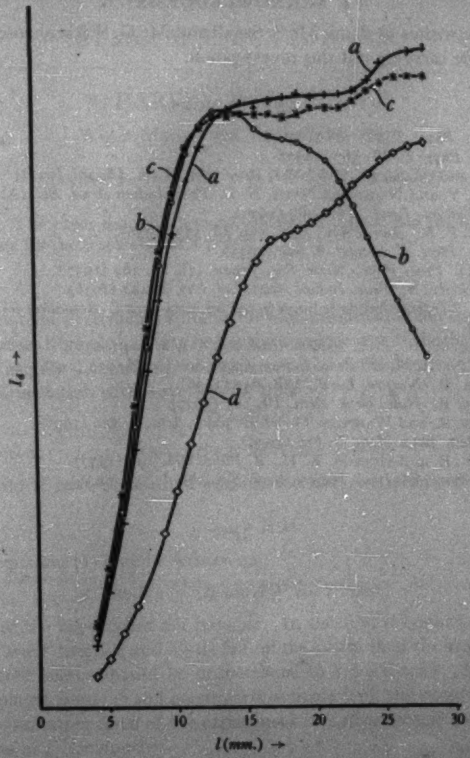

Figure 6. Dependence of the intensity of diffracted light on l for fixed values of α under operating conditions resulting in curves of the type shown in figure 1 with a central minimum (a, b, c) and without one (d).

Extermann and Wannier⁽¹⁵⁾, p. 256), there are slight indications of a periodicity in the form of systematic unevennesses. An investigation of separate orders of diffraction will presumably show such periodicities more markedly.

§ 4. ACKNOWLEDGEMENT

The author wishes to thank Mr S. Sagall and Mr G. Wikkenhauser for granting the facilities for carrying out this investigation.

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THE DISPERSION OF A PRISM

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ABSTRACT. The paper describes a method, suitable for use and teaching, of verifying Monk's formula for the dispersion of a prism.

The work to be described is suitable for honours students but the only reference to it that we have been able to find is the development of the theory by Monk(1).

The angular dispersion D of a prism may be defined as the rate of change of the angle θ of deviation with wave-length, thus

$$D = d\theta/d\lambda. \qquad \ldots\ldots(1)$$

Monk shows that

$$D = -2B \sin A/\lambda^3 \cos i' \cos r, \qquad \ldots\ldots(2)$$

where i', r and A have their usual significance and B is obtained from Cauchy's formula

$$\mu = \mu_0 + B/\lambda^2. \qquad \ldots\ldots(3)$$

Obviously, equation (1) may be written as

$$d\theta/d\lambda = -2B \sin A/\lambda^3 \cos i' \cos r. \qquad \ldots\ldots(4)$$

Two forms of the experiment are possible. In the first it could be shown that if radiations of wave-length λ and $\lambda + d\lambda$ fall on the prism then the angular separation of the refracted beams should be proportional to $1/\cos i' \cos r$. Alternatively, $d\theta$ may be calculated from (4) and a comparison made with the observed values.

To carry out either form of the experiment it is necessary to express $\cos i'$ and $\cos r$ in terms of i. Obviously

$$\cos r = \left(1 - \frac{\sin^2 i}{\mu}\right)^{\frac{1}{2}}. \qquad \ldots\ldots(5)$$

The evaluation of $\cos i'$, however, presents a little more difficulty:

$$\sin i' = \mu \sin r' = \mu \sin (A - r)$$

$$= \mu \sin A \cos \left(\sin^{-1} \frac{\sin i}{\mu}\right) - \cos A \sin i, \qquad \ldots\ldots(6)$$

whence i' and hence $\cos i'$ are known.

Equations (2) and (3) are evaluated separately, the mean value for the prism for the red and blue hydrogen lines being used for μ, and the experimentally determined value for the angle of the prism for A. This is done for convenient values of i from 50° upwards. The two results for each value of i are then combined and the reciprocal multiplied by $2B \sin A\, d\lambda/\lambda^3$, where B is obtained from Cauchy's formula (with the values of μ_R and μ_B already obtained), $d\lambda$ is the difference in wave-length between the red and blue hydrogen lines, and λ is the mean wave-length for these two lines. The final result is then shown graphically, $d\theta$ being plotted against i.

$d\theta$ is then observed for these two lines for various values of i by means of an ordinary student's spectrometer. The results obtained from such an experiment are shown in the table, as also are the calculated values of $d\theta$ read from the graph for the particular values of i employed.

i	52° 10′	54° 0′	57° 10′	62° 10′	68° 2′	72° 10′	77° 10′	81° 10′
$d\theta_{\text{obs.}}$	6 30	5 29	4 19	3 30	2 55	2 45	2 36	2 27
$d\theta_{\text{calc.}}$	6 36	5 12	4 4	3 23	3 2	2 52	2 43	2 39

It will be seen that the agreement between the theoretical and observed values is quite good; better, in fact, than might have been expected from the fact that the mean value of λ which has to be employed is well removed from the λs actually used. $d\lambda$, however, cannot be made a great deal smaller, because then $d\theta$ will become so small as to be measurable only with great difficulty and not with any considerable accuracy.

We have to thank Mr H. Bell of the University of Manchester for discussing this problem with us.

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THE EXPERIMENTAL VERIFICATION OF THE DIFFRACTION ANALYSIS OF THE RELATION BETWEEN HEIGHT AND GAIN FOR RADIO WAVES OF MEDIUM LENGTHS

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ABSTRACT. Some {height, gain} curves obtained by von Handel and Pfister from observations taken in an aeroplane on medium-wave broadcasting stations show that initially there is a diminution of signal-strength on going up from the ground. This feature is predicted by the complete diffraction theory of propagation from a vertical aerial over an imperfectly conducting curved earth, and can be represented by a simple analytical expression. It is also a consequence of the flat-earth theory, and the analytical expression of this initial drop is shown to be of exactly the same form for a flat earth as for a curved earth. As Norton has shown, in the flat-earth case the initial drop may be explained physically as an interference effect between a space wave and a surface wave, and the equivalence of the {height, gain} relation for small heights suggests that in the curved-earth case also the total signal strength is the resultant of a space wave and a surface wave, both reaching to points beyond the horizon by diffraction, and suffering the same attenuation relative to their flat-earth values. The theory shows that though the initial drop is negligible on very short and on very long waves, it should be observable in practice under the conditions in which the medium-wave tests were carried out. A {height, gain} curve for a value 10^{-13} e.m.u. of σ is calculated from the diffraction theory for comparison with the experimental curves of von Handel and Pfister, and very good agreement is obtained. The span of the initial drop and the slope of the eventually rising curve change rapidly as σ is altered, and it is suggested that aeroplane experiments on medium waves should form a useful method of determining the mean value of the conductivity σ of the earth over any given ground.

Some aeroplane experiments carried out on medium waves in Germany by von Handel and Pfister confirm in a striking way the {height, gain} analysis derived from the complete diffraction theory of propagation over an imperfectly conducting earth. This theory has been worked out independently by three methods(1, 2, 3) which are essentially equivalent, and which agree in giving the rather unexpected result that on going up from the surface of the earth there is an initial diminution of signal-strength.

By computing a particular case, Wwedensky(1) showed that this initial drop is also obtained according to flat-earth theory, and this feature has been discussed more fully by Norton(4). He has given the expression for the electric field at a point above an imperfectly conducting flat earth at a distance from a vertical half-wave transmitting dipole as the vector sum of two separate fields, and this resolution is more than an analytical device, since it is possible to give a simple physical interpretation of the two waves which would produce these fields.

One of the fields is zero at the surface of the earth, and initially it increases linearly with height. It corresponds to a space wave E_{sp}, which for large numerical distances may be regarded as the effect of the aerial and its image in the imperfectly conducting earth. The other field, which represents the whole field at the surface of the earth, initially decreases by an amount proportional to the square of the height. It corresponds to a surface wave E_{su} possessing the general characteristics of a Zenneck wave, allowance being made for the fact that it is not strictly a plane wave but originates from a vertical dipole. In particular, Norton has shown that its electric vector has the typical forward tilt, and that it supplies all the energy to the ground currents.

The addition of a wave which decreases with the height to one which increases from a zero value at the surface of the earth, must produce an initial drop in field strength, but the analysis shows that the resulting minimum is greatly accentuated, or even over-ruled, by the fact that the space wave is initially out of phase with the surface wave and can partially interfere with it. This phase interference was pointed out by Norton from his graphs, and it can be exhibited in a simple analytical form by expanding his expressions for E_{su} and E_{sp} in terms of the height h.

If we assume that the numerical distance is large, say greater than 40, and that the slight forward tilt of the electric vector of the surface wave can be neglected, then initially for small values of h the surface wave E_{su} remains effectively equal to the value, say E_0, which it has at the surface of the earth, and it can then be shown that the total field E_h, equal to $E_{su} + E_{sp}$, at the height h is given by

$$\frac{E_h}{E_0} = 1 + \frac{h}{\zeta}\frac{2\pi}{\lambda}\,e^{\iota\pi/2},$$

where

$$\zeta = \frac{\epsilon - \iota 2\sigma\lambda c}{\sqrt{(\epsilon - 1 - \iota 2\sigma\lambda c)}}$$

for an assumed time factor $\exp(\iota\omega t)$. ϵ is the dielectric constant of the earth, and σ is the conductivity of the earth measured in e.m.units, while c is the velocity of light and λ the wave-length.

Now we have shown (3) that exactly the same relation is obtained from the complete diffraction theory for a curved earth. We discussed this relation in detail for the particular case in which the first term of the diffraction formula is predominant, but, as the relation is independent of the order of the term, a little consideration shows that it is still true in the general case when several terms may have to be taken. The exact equivalence of the two theories in this respect suggests that the field, even for points far below the line of sight, can be considered as the sum of a surface wave and of a space wave diffracted down from above, which can be derived from the corresponding flat-earth waves by the application of the same attenuation factor representing the effect of the earth's curvature. The form of this correction factor as a function of the distance from the transmitter has been discussed by van der Pol and Bremmer(2).

It is an interesting feature of the two waves that for small heights the ratio of their amplitudes is independent of the distance from the transmitter, while the interference effect is independent of the wave-length in the sense that there is no periodic variation in and out of phase, but only a gradual unidirectional change of the phase angle between them from $\pi/2$ to $3\pi/4$, as the wave-length increases, and the phase angle of $1/\zeta$ increases from 0 to $\pi/4$.

We have given a simple vectorial representation of the value of E_h/E_0 as the addition of the vector

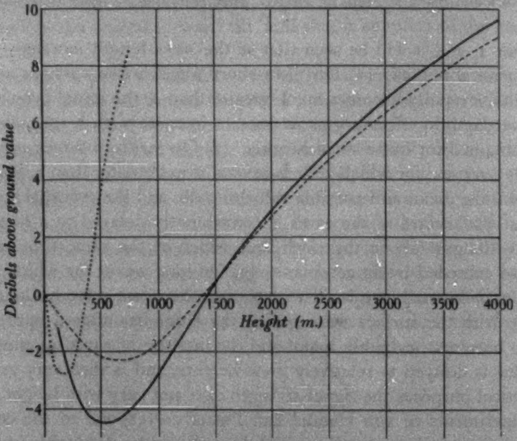
$$\frac{h}{\zeta}\frac{2\pi}{\lambda}e^{\iota\pi/2}$$

to the vector 1, and it will be seen that as the wave-length increases we have the following general results. (1) On ultra-short waves when $\epsilon \gg 2\sigma\lambda c$ and, even for small heights, h rapidly becomes much greater than λ, the initial drop is negligible; the signal-strength in effect begins at once to increase with height, and very large gains are obtained for quite small heights. (2) On medium waves overland, or on short waves over sea, for which $2\sigma\lambda c$ is several times greater than ϵ, the initial drop can approach the maximum possible value of 3 db. and the eventual increase above the value at the surface of the earth is conveniently delayed to a height which is several wave-lengths above the earth, but which at the same time can easily be attained and exceeded by an aeroplane. (3) On long waves for which $2\sigma\lambda c \gg \epsilon$, for small heights h is small compared with λ, and the space wave no longer becomes comparable with the surface wave before the latter has altered appreciably. The initial drop becomes negligible again, and the increase of signal-strength above the surface value is delayed to relatively great heights, and is then very gradual. Thus for all practical purposes the signal-strength does not vary with height.

The experiments of von Handel and Pfister correspond to the condition (2), where the theory predicts that the initial drop should be detectable in practice. In figure 21 of their paper, von Handel and Pfister(5) give a curve obtained from measurements made in an aeroplane flying at a distance of 150 km. from the Leipzig transmitter, radiating on a wave-length of 382 m., and in figure 13 of a later paper, Pfister(6) has given a similar curve obtained from measurements made at Munich of the signal-intensity produced by a transmitter at Berlin 501 km. away, radiating on a wave-length of 405·4 m. Both of these curves show a very marked initial drop which delays the increase above the surface value to a height of 1500 m., and this suggests that they should be compared with the theoretical curves computed from the diffraction theory.

As the wave-lengths are of the same order and the curves are so similar, we need only to consider one of them, and in figure 1 the curve given by Pfister is reproduced on a decibel scale, with the theoretical curve computed from the complete diffraction theory for the case in which $\epsilon = 5$ and $\sigma = 10^{-13}$ e.m.u. for comparison. It will be seen that the agreement is remarkably good. Actually, the experimental curve as originally plotted has been shifted up 2 db. to make the two curves agree where they rise after the initial drop. This was justifiable, as the value of the signal-strength at the surface of the earth was used as datum in converting the experimental curve to a decibel scale, and we should not expect this value to be as reliable as those taken well above the surface.

This shift only produces a small change in the span of the initial drop, and it brings the curves together at the point where the comparison of the slopes is the really significant feature. The experimental curve shows a rather greater initial drop than the theoretical curve. This may be due to the disturbing effect of ground irregularities on measurements made near to the surface of the earth, but of the existence of the drop the experiments leave no doubt, and the curves show a remarkable agreement, both as regards the span of the initial drop and the slope of the eventually rising curve.



Wave-length, 405·4 *m.*

Figure 1. —— Experimental curve by W. Pfister; – – – Theoretical curve for $\epsilon=5$, $\sigma=10^{-13}$ e.m.u.; ….. Theoretical curve for $\epsilon=5$, $\sigma=10^{-14}$ e.m.u.

Besides confirming the theory, the experiments suggest a new method of determining the conductivity of the earth. The theory shows that the span of the initial drop is approximately proportional to $\sqrt{\sigma}$, when $2\sigma\lambda c \gg \epsilon$, and that the slope of the eventually rising curve increases rapidly as σ is decreased. In illustration of this, the theoretical curve for the case in which $\epsilon=5$ and $\sigma=10^{-14}$ e.m.u., is also shown in figure 1. This curve was not computed in such detail as the one for $\sigma=10^{-13}$ e.m.u., but it serves to show that the experiments decide in favour of a value very close to $\sigma=10^{-13}$ e.m.u. for the ground concerned; the original choice in computing the theoretical curve happened to be a very fortunate one for making the comparison.

Since the {height, gain} curve is so susceptible to small changes of σ, it appears that a set of experiments made with an aeroplane on various wave-lengths to study the {height, gain} relation, would provide a most useful way of determining the conductivity accurately. It may be asked how aeroplane experiments can measure the conductivity of the earth if this varies from place to place. Do they, for instance, measure the conductivity of the earth in the neighbourhood of the aeroplane, or the mean conductivity along the whole route? These questions cannot be answered until we have a theory in which account is taken of the variation of conductivity. Such a theory would inevitably be much more complex than the already sufficiently complex theory for uniform conductivity. One thing seems certain, that lack of uniformity of conductivity should be shown up in the comparison of the observed and calculated {height, gain} curves. The observed curve can hardly be of the correct form for uniform conductivity if the conductivity varies along the path.

The agreement of theory and observation will be a sufficient guarantee that (1) the conductivity is at least approximately constant along the path and that (2) the conductivity so deduced will represent with sufficient accuracy the mean value along the path.

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A MECHANICAL MODEL ILLUSTRATING THE PRINCIPLE OF THE CYCLOTRON

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ABSTRACT. The model illustrates by a mechanical analogy the behaviour of the ions accelerated within a cyclotron. Two horizontal semicircular brass plates oscillating in opposite phase in a vertical direction represent the dees of the cyclotron; the plates are connected by a hinged ramp, and a steel ball representing an ion rolls round a series of semicircular grooves cut on the upper surfaces of the plates and receives repeated accelerations by rolling down the ramp, its path being the same as that described by an ion in the cyclotron under the combined influence of the electric and magnetic fields.

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§1. INTRODUCTION

THE cyclotron appears likely to play such an important part in atomic physics that it seems desirable for instructional purposes to design a mechanical model which shall illustrate in a simple way its mode of operation. The present model has been constructed in the Science Museum workshops for exhibition by the side of the early cyclotron constructed by Lawrence(1) in 1931, which he has kindly lent to the Museum. The model as here described is suitable for lecture demonstration, but for museum use it will need to be fitted with some form of automatic feed device so that it may be continuously operable by the public.

§2. GENERAL DESCRIPTION

As shown in the photograph (figure 1) the model consists essentially of two horizontal semicircular brass plates, each of radius 14 cm., connected by a hinged ramp. Each plate is mounted upon a brass sleeve which slides over a vertical steel column. One of the plates is forced to execute a vertical harmonic motion of amplitude about 0·2 cm. by means of a connecting-rod driven from an eccentric of variable stroke, while the other plate is caused to describe an identical harmonic motion of exactly opposite phase by means of a symmetrical connecting system. Provision is made for the necessary slight horizontal slip of one of the plates.

The level surfaces of the plates represent the regions of constant electric potential within the dees of the cyclotron, and their harmonic rise and fall correspond to the harmonic variations of potential of the dees due to the impressed radio-frequency electric oscillations, while the slope of the connecting ramp represents the electric gradient which accelerates the ions, the difference of level of the two plates at any instant being equivalent to the difference of electrical potential between the dees.

In the cyclotron itself any ion released in the neighbourhood of the filament is caused, by the combined influence of the uniform magnetic field and the oscillating electric field between the dees, to describe a series of semicircles of continually increasing radius, the time required to cover each semicircle being the same and independent, within wide limits, of the position from which the ion starts and the phase at which it is released. The mechanical model cannot reproduce this general resonance property, but it illustrates the characteristic mode of operation of the cyclotron and gives an accurate representation of the path in space and time pursued by selected ions released under certain initial conditions.

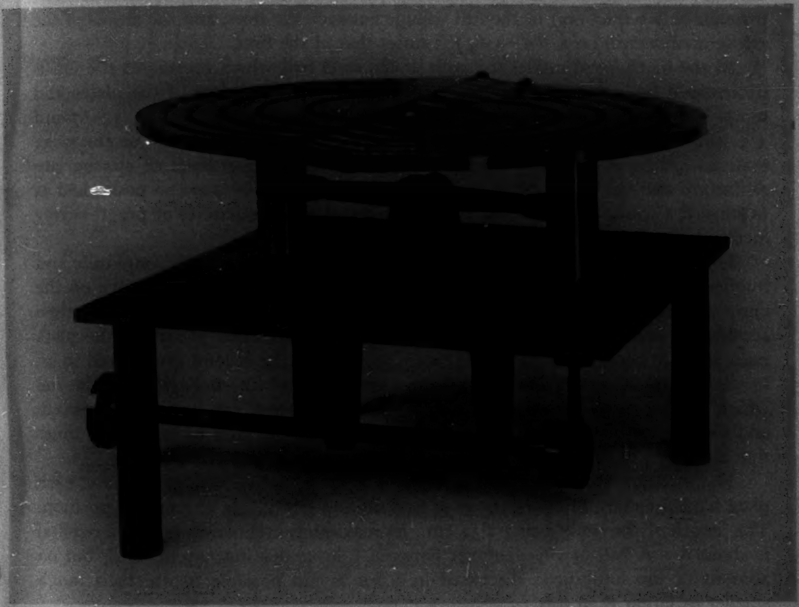

Figure 1. General view of the model.

Consider a positive ion released near the mouth of one of the dees at an instant at which its positive potential is V electrostatic units above that of the other dee. It is attracted across to the other dee and acquires a velocity v_1 given by $\frac{1}{2}mv_1^2 = Ve$, where e and m are respectively the charge and mass of the ion; v_1 is therefore equal to $\sqrt{(2Ve/m)}$.

The ion then describes a semicircle within the dee and, if the electric oscillations and magnetic field are in proper adjustment for resonance, the ion will on its emergence find the difference of potential between the dees again equal to V, but in opposite sense, so that it will acquire a further increment Ve of energy and its velocity will rise to a value v_2 equal to $\sqrt{(4Ve/m)}$.

After n accelerations its velocity will be $\sqrt{n}\,\sqrt{(2Ve/m)}$, so that the velocity increases proportionally to $\sqrt{n}$. Since the radius of each semicircle is proportional to the velocity with which it is traversed, the radius of the nth semicircle is $\sqrt{n}$ times that of the first one. This relationship holds for all ions released near the mouth of either dee. An ion released at any intermediate position will at its first acceleration fall through only a fraction f of the full voltage between the dees, and the radius of the nth semicircle will be $\sqrt{\{(n-1+f)/f\}}$ times that of the first.

In the mechanical model the path of the steel ball which represents the ion is determined by a series of grooves turned in the upper surface of the plates and ramp. One plate bears grooves whose radii are proportional to 1, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ and $\sqrt{9}$, while the radii of the grooves in the other plate are proportional on the same scale to $\sqrt{2}$, $\sqrt{4}$, $\sqrt{6}$, $\sqrt{8}$ and $\sqrt{10}$. Short connecting grooves of appropriate radius are also cut in the ramp and the three series of grooves are so positioned as to furnish a continuous path for the ball in a series of semicircles of progressively increasing radius.

To operate the model, the plates are set oscillating at a suitable amplitude and frequency and the ball is placed at the beginning of the innermost groove on the ramp and released at the moment at which the slope is steepest; the ball then accelerates down the slope, runs round the first semicircular groove as the plate rises, is again accelerated down the ramp, runs round the second groove and so on to the outermost semicircle, reproducing very closely[*] the behaviour of the ion which it represents. The gain in velocity at each descent can be easily observed, and the velocity attained in the outermost semicircle is seen to be several times that attained in a single acceleration.

The model works best at a frequency of 33 to 35 oscillations per minute, each plate having an amplitude of travel of 0·18 cm. on either side of its mean position. The length of the ramp being 2·54 cm., its maximum inclination to the horizontal is about 8°. A ball of any diameter between $\frac{3}{8}$ in. and $\frac{5}{8}$ in. may be used, but on account of the limitations discussed in § 5 it is not possible to use balls much smaller or larger than this.

§ 3. THE DESIGN AND CONSTRUCTION OF THE PLATES AND THEIR GROOVES

Figure 2 shows the lay-out of the groove system. As has been mentioned above, it is so arranged that the ball describes a series of successive semicircles whose radii are proportional to the square roots of the natural numbers; for convenience in construction these radii are rounded off in multiples of $\frac{1}{16}$ in. The plates A and B, together with the ramp C, form a circular disk, 11 in. in diameter, of which O is the centre. The edges of the plates and ramp are flattened at DE and FG for

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[*] The first groove on the ramp does not quite accurately reproduce the initial path of an ion as it starts from rest.

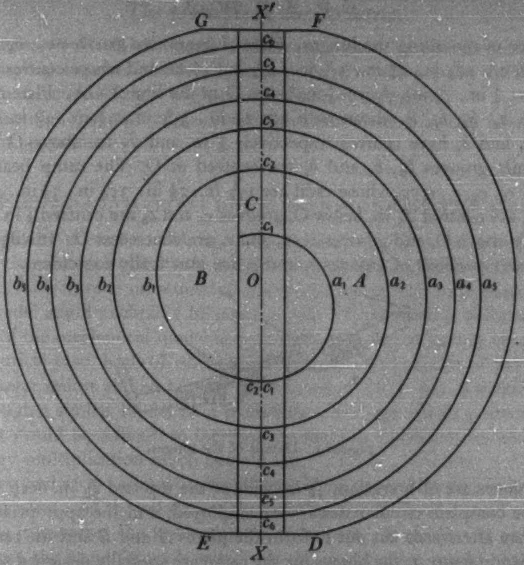

Figure 2. Lay-out of the groove systems.

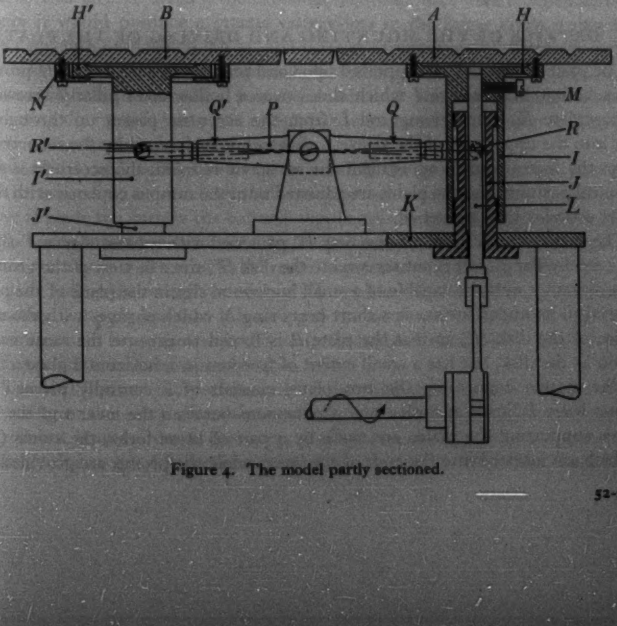

Figure 4. The model partly sectioned.

convenience in mounting the hinges. Plate *A* bears five grooves a_1, a_2, a_3, a_4, a_5, whose radii are $1\frac{9}{16}$ in., $2\frac{3}{4}$ in., $3\frac{9}{16}$ in., $4\frac{3}{16}$ in., $4\frac{3}{4}$ in. and whose centres are respectively $\frac{9}{16}$ in., $\frac{3}{8}$ in., $\frac{5}{16}$ in., $\frac{5}{16}$ in., $\frac{1}{4}$ in. below *O* in the line *X′OX*. Plate *B* bears five grooves b_1, b_2, b_3, b_4, b_5 whose radii are $2\frac{1}{4}$ in., $3\frac{3}{16}$ in., $3\frac{7}{8}$ in., $4\frac{1}{2}$ in., and 5 in. Grooves b_1 and b_2 have centres respectively $\frac{1}{8}$ in. and $\frac{1}{16}$ in. above *O* in the line *X′OX*, while grooves b_3, b_4 and b_5 are centred at *O*. The ramp bears pairs of grooves c_1, c_2, c_3, c_4, c_5, c_6 whose radii are $1\frac{9}{16}$ in., $2\frac{1}{4}$ in., $3\frac{3}{16}$ in., $3\frac{7}{8}$ in., $4\frac{1}{2}$ in., 5 in. Grooves c_1 are centred $\frac{9}{16}$ in. below *O*, grooves c_2 and c_3 are centred $\frac{1}{8}$ in. and $\frac{1}{16}$ in. respectively above *O*, and grooves c_4, c_5 and c_6 are centred at *O*. It will be noticed that the lower portions of grooves c_1 and c_2 are practically coincident.

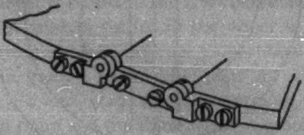

Figure 3. Details of the hinges.

The grooves are of **V** section, $\frac{3}{16}$ in. wide at the top and $\frac{3}{32}$ in. deep; in turning them, three complete circular discs were employed, and the appropriate portions of these were afterwards cut out to form the plates *A* and *B* and the ramp *C*.

As shown in figure 3, the hinges for the ramp are specially designed so that their axes of rotation lie in the plane of the surface, and the adjoining edges of the plates and ramp are bevelled off as shown in figure 4 to permit their relative motion.

§ 4. DETAILS OF THE MOUNTING AND DRIVING OF THE PLATES

The plate *A*, figure 4, is supported upon and screwed to a brass disk *H* provided with a hollow brass sleeve *I* which slides over a hollow steel pillar *J* screwed to the baseplate *K*. The driving rod *L* from the eccentric passes up through this pillar into the upper part of the sleeve *I*, to which it is clamped by the set screw *M*. To set the instrument in adjustment the set screw is freed, the eccentric is set in its central position, and the plates are adjusted until the ramp is coplanar with them: the set screw is then tightened.

The plate *B* rests upon a brass disk *H′* provided with a brass sleeve *I′* sliding over a steel pillar *J′*, but is not screwed to the disk *H′*, since in view of the geometry of the system it must be capable of a small horizontal slip in the plane of the paper. Screwed to its under surface is a short brass ring *N* which engages with the under surface of the disk *H′*, so that the plate *B* is forced to execute the same vertical motion as the disk, but has a small extent of freedom in a horizontal plane.

The system connecting the two plates consists of a centrally pivoted steel rocking lever *P* bored at both ends. Connexions between the lever and the brass sleeves supporting the plates are made by a pair of brass forks, the stems *Q*, *Q′* of which are inserted into the ends of the lever, while the prongs are provided with pairs of screws *R, R′* which engage with bearing holes in the sleeves. The forks, besides coupling together the two halves of the system, prevent any slip of the plate *B* in a direction at right angles to the plane of the paper.

It is essential that the whole instrument should be properly levelled before being set in operation.

§ 5. FACTORS LIMITING THE DIMENSIONS OF THE MODEL

Although the present model, in which the ball describes ten semicircles, well illustrates the characteristic mode of operation of the cyclotron and is of a convenient size for lecture or museum demonstration, it is of interest to enquire whether a larger model could profitably be constructed. Unfortunately, however, the performance of the mechanical model is limited by two factors having no counterpart in the electrical case, both of which tend to reduce the velocity of the ball in the paths of larger radius and so to spoil the resonance effect, and in practice it seems that, although a similar model on a uniformly larger scale could probably be made to work, it would be difficult to increase the number of semicircles and hence the total energy-multiplication much beyond ten.

The first and most serious limiting factor is, of course, the fact that the motion is not frictionless, and the velocity of the ball therefore falls off instead of remaining constant as it traverses the level portions of its path. If the loss of energy due to friction were the same for each semicircle the isochronism would not be spoiled, since the loss would be a constant fraction of the gain down each slope; but it is easily seen that friction produces a greater loss of energy in the larger than in the smaller circles. Even if the frictional retarding force were independent of the velocity it would produce a greater energy-loss in the larger circle owing to the greater distance over which it acts, and since the friction almost certainly increases with velocity the effect is correspondingly accentuated. In fact, as the radius of the semicircle is increased, a point is reached at which the energy gained in descending the ramp is just equal to that lost in friction in traversing the semicircle, so that no further acceleration is possible.

In order to reduce friction as far as possible the grooves are made narrow, so that the two points at which the ball is supported approach each other, but if the groove is made too narrow the ball will topple over and escape from it at the higher speeds owing to centrifugal force. In the present model the groove is made $\frac{3}{16}$ in. wide at the top, a width which theoretically permits a maximum rate of 53 c./min. for a $\frac{1}{2}$-in. ball on the outermost circle of 5 in. radius. The maximum speed attainable in practice is a little less than this owing to the fact, mentioned below, that the ball does not remain in contact with the ramp throughout its descent. For a $\frac{3}{8}$-in. ball the theoretical maximum speed is 47 c./min.

A second limit to the performance of the mechanical model is set by the fact that owing to the sharp discontinuity of slope at the line of hinges, the ball does not roll but bounces down the ramp, and so does not receive the full increment of energy due to the difference in level of the two plates. This effect again is most troublesome in the outer portions of the ball's path, where it is moving more rapidly, and calculation shows that with the model oscillating at about 35 c./min. the ball descends the outermost ramp in a few bounces, the actual number depending upon the amplitude of the oscillations. On account of the bouncing effect, increase of amplitude, which reduces the number of bounces, produces relatively little increase in the frequency at which the model best operates.

§ 6. ACKNOWLEDGEMENTS

I wish to express my thanks to Col. E. E. B. Mackintosh, D.S.O., Director of the Science Museum, for permission to publish this paper. I also wish to acknowledge the valuable assistance of Mr C. A. Duddles, to whom is due the detailed design of the hinges and the mechanism for supporting and driving the plates.

REFERENCE

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THE EFFECT OF ADAPTATION AND CONTRAST ON APPARENT BRIGHTNESS*

By F. H. G. PITT

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ABSTRACT. Measurements have been made of the variation of the brightness-sensitivity of the eye over a range of adaptation brightnesses extending from 0·001 to 400 equivalent foot-candles. The method adopted involves the use of binocular matching, the left eye, at a constant state of adaptation, being used as a reference standard, and the right eye, adapted to various brightnesses, viewing a test patch. From the experimental results, curves each of which represents physical brightnesses differing from those for subjective black by a constant number of just noticeable brightness-differences, and curves of equal apparent brightness have been constructed. It is shown that the brightness of the surround field has a very great effect on the apparent brightness of the test patch, the change caused by changing the surround brightness being almost instantaneous. The effect of this surround field, viewed with the right eye, on the comparison patch, as viewed with the left eye, both fields merging binocularly, is discussed and is shown to be negligible at least in comparison with the first-mentioned effect. The results obtained show that it may be impossible to construct sensation curves of the simple type hitherto postulated for considering the reproduction of photographic tone.

§ 1. INTRODUCTION

The work to be described in this paper arose from a consideration of the problem of the reproduction of tone in photography. The view has been taken that the reproduction of the tones of the subject photographed should be subjectively correct; that is to say, the impression given by the print should be identical with the impression given by the subject, under their respective conditions of viewing. In general, the adaptation brightness of the eye is much lower for viewing a print than for viewing the subject, although the converse sometimes occurs. As the sensitivity of the eye varies considerably with its state of adaptation, a knowledge of the variation of sensitivity with adaptation is fundamentally necessary for consideration of tone-reproduction in the above sense.

In the only direct attack so far on this particular problem, use has been made of sensation curves. These sensation curves have usually been constructed by integration of the contrast sensitivity curves. Abribat(1) used such curves, measured by him under conditions comparable with those obtaining in practice, to construct a theory of subjective tone reproduction. Using the equation

$$S = \int \frac{B}{\Delta B}\, d(\log B),$$

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* Kodak Communication No. H 688 from the Research Laboratory, Wealdstone, Middx.

where S is the sensation and B the brightness, he derived sensation curves for various conditions of adaptation from measurements of $B/\Delta B$, the inverse of the Fechner fraction. Given the adaptation conditions and corresponding sensation curves for the eye when the scene and reproduction are viewed, he showed graphically, by a method due to Jones(2), how the brightnesses of the former must be reproduced in the latter. Briefly, the method is as follows. Quadrants 1 and 2, figure 1, contain the sensation curves appropriate to the states of adaptation of the eye when the scene and reproduction respectively are being viewed, the subjective blacks of both being on the same horizontal line. Quadrant 3 contains a line drawn at 45°. The curve shown in quadrant 4 is obtained by orthogonal projection; that is, by tracing lines similar to *ABCD* and *AD*. The curve in quadrant 4 then represents the objective brightnesses of the picture as a function of the objective brightnesses of the subject required to give correct subjective reproduction.

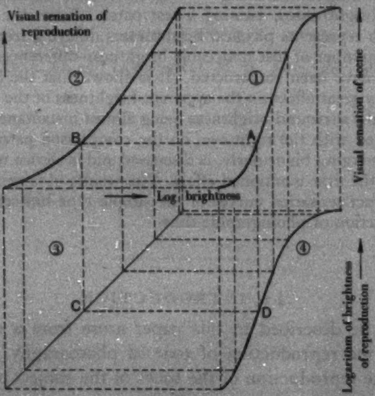
Visual sensation of reproduction
②
①
B
A
Log brightness
Visual sensation of scene
③
④
C
D
Logarithm of brightness of reproduction

Figure 1.

Whether or not it is possible to measure sensation is a problem that has never been settled. Recent papers(3) suggest that a sensation scale cannot exist, while methods which purport to measure these values must necessarily have large experimental errors. The question of sensation has been discussed at some length by Wright(4). The viewpoint of the present work is that it may be possible to construct a theory of tone-reproduction which avoids the use of sensation curves entirely. The sensation curves are regarded as artificial constructs from experimental observations.

It is necessary at this stage to define three terms which will be constantly used. If we consider that an observer is viewing a particular object, such an object will have a brightness which can be measured in physical units and this brightness will be referred to as the *physical brightness*. The eye of the observer may be regarded as being adapted to a particular brightness, and this brightness will be referred to as the *adaptation brightness*. Owing to the sensitivity of the eye varying with the adaptation brightness, the apparent brightness of the object, to the observer, will vary according to the physical brightness of the object viewed and to the adaptation brightness. The term *apparent brightness* will be used with this connotation.

Let us suppose that there is some method of recognizing a given apparent brightness, whatever the adaptation brightness may be, so that it becomes possible to make a set of curves by which the relation between physical brightness and adaptation brightness is represented, and such that each curve is a line of constant apparent brightness. Such curves, plotted for convenience in logarithmic units, are shown in figure 2, in which the ordinates measure the physical brightness, and the abscissae the adaptation brightness. From such a graph it would be evident that

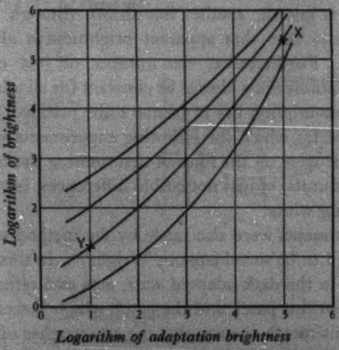

Figure 2. Curves of equal apparent brightness.

an observer would regard two surfaces as of the same apparent brightness if in one case the logarithm of the adaptation brightness were 5·0 and the logarithm of the physical brightness were 5·4, as at point *X* in figure 2, and in the other the logarithm of the adaptation brightness were 1·0 and the logarithm of the physical brightness were 1·2, as at point *Y*, for both points are on the same apparent brightness curve. It is obvious that graphs of this kind, covering a sufficient range, would enable one to calculate the correct tone-reproduction curve for any two specified adaptation brightnesses, and would overcome the difficulty of using sensation curves, which according to many people cannot be measured.

Curves of the kind shown in figure 2 are already available for two special and limiting types of sensation, subjective black and subjective blinding white. The former is the lower threshold value of physical brightness for any particular adaptation brightness, any lower physical brightness being visually indistinguishable from it. Measurements of this quantity can be made with a certain amount of facility. The latter is the upper threshold, for any particular adaptation brightness, but owing to such high physical brightnesses having a marked effect on the adaptation of the eye, measurements of this quantity are much more difficult to make. The requisite measurements have been made over a large range of adaptation brightnesses by Nutting⁽⁵⁾ and have been used by Hopkinson⁽⁶⁾ to construct a scheme of tone reproduction.*

A sensation intermediate between subjective black and subjective blinding white cannot be specified in the same way as the above two limiting sensations. It may, however, be regarded as differing from subjective black or subjective blinding white by a certain number of just noticeable increments of physical brightness, the size of the steps varying according to the experimental conditions, such as field-size, etc. Such a statement at once presupposes that the number of just noticeable differences between subjective black and subjective blinding white is equal for all adaptation brightnesses, as may readily be seen by the argument that if one line of constant apparent brightness is known, another line drawn through the points which are just noticeably brighter than that apparent brightness is also a line of constant apparent brightness. Furthermore, the number of just noticeable differences between any two sensations must always be constant for all adaptation brightnesses. The validity of this assumption has of course to be justified, and justification of it was one of the reasons for which the following experiments were made. Thus, the first experiments made were on the lines of determining the number (and amounts in physical brightness units) of just noticeable differences, between subjective black and subjective blinding white.

In addition, experiments were also made by the method of binocular matching which has been proved to be so valuable by Wright⁽⁷⁾. In this method, the left eye, which is always kept in the dark-adapted state, acts as a reference standard, and a match is made between this patch and the patch which is seen in the right eye, by varying the latter. This match ensures that the two patches of light are of the same apparent brightness. By altering the brightness of the patch as seen by the left eye, and by varying the adaptation brightness of the right eye, curves of the kind shown in figure 2 can be plotted over the necessary range.

§ 2. OPTICAL SYSTEM OF APPARATUS USED

A diagrammatic representation of the optical lay-out is shown in figure 3. A Pointolite lamp is placed at the focus of the achromatic lenses L_1 and L_2, and the two parallel light beams, after passing through neutral non-scattering wedges W_1 and W_2, are brought together by means of a photometric prism. The light from this prism, after passing through the non-scattering neutral filters F_1 and F_2, is brought to a focus by the achromatic lens L_3 at a point E_1. Thus, when the eye is placed at E_1, a simple bipartite field is seen, and the angular subtense, limited by the aperture in the adapting screen, is about 2°. In an investigation of this type the scene or picture should be viewed with the unaided eye.

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* It should be pointed out that Davies, in the discussion, has shown that Hopkinson's results can be obtained by an argument similar to that of Abribat only if the sensation curves are parallel.

With these facts in mind, it is felt that the use of an artificial exit pupil at E_1 would not conform with the required conditions, and consequently the only pupil, at E_1, is the observer's own eye pupil. The adapting screen, coated with magnesium oxide, serves as a surround field, and subtends at the eye an angle of about 90°. This screen, illuminated by means of the adapting lamp, may be varied by using different lamps and lamp-distances. The eye views this adapting field for a sufficient time to reach the equilibrium state of adaptation, and in the experimental results the state of adaptation is specified by the brightness of the screen. The two wedges W_1 and W_2, calibrated by means of sectors, have a density range of 0 to 3·0, while the maximum density of each of the filters F_1 and F_2 is 4·0. It is therefore possible to insert a range of densities from 0 to 11·00 in the optical system, thus cutting down the light falling on E_1 from 1 to 10^{-11} if necessary.

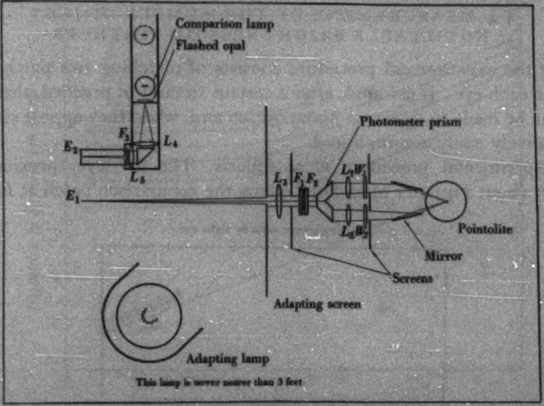

Figure 3.

It has been pointed out that measurements of blinding white, owing to the high brightnesses having a marked effect on the adaptation of the eye, are only made with a certain amount of difficulty. It was also recognized that measurements as far as blinding white were for our purpose unnecessary, and so a certain upper limiting value was set. To do this, a second part of the apparatus was necessary. This consists of a comparison lamp which illuminates a thin flashed opal glass, placed at the focus of the lens L_4, and the distance of the lamp from the opal may be varied, with consequent variation of the surface brightness of the opal glass. Parallel light passes through the prism and is brought to a focus at an artificial exit pupil E_2 having a diameter of about 2 mm. The observer's left eye views the opal surface, which is suitably apertured to give a 2° field. The physical brightness of this surface can be set to any particular value, and acts as a reference standard and as the upper limiting value just mentioned. This patch of light is known as the comparison patch.

The distance between E_1 and E_2 can be varied to suit the interocular distance of the observer, and as it is important that the head of the observer should be rigidly fixed during the experiment, the method used by Stiles and Crawford(8) was adopted. In this case, dental modelling composition is used to form an impression of the observer's teeth. The hardened impression is placed in the required position so that on biting into it, the observer fixes his head very effectively.

The voltages across both lamps, measured by voltmeters, are kept constant by means of variable resistances.

§ 3. MEASUREMENT OF THE NUMBER OF JUST NOTICEABLE BRIGHTNESS-DIFFERENCES

Part of the experimental procedure consists of matching two patches of light, seen one in each eye. It is found, after a certain amount of practice, that these two patches can be made to appear in juxtaposition and, when they appear so, matching is comparatively easily accomplished.

The experimental procedure is as follows. The left eye, previously dark-adapted for about a quarter of an hour, views the comparison patch at E_2, which is of constant physical brightness. As this eye is adapted to a fixed adaptation brightness, being dark-adapted, it follows that the apparent brightness of this comparison patch also remains constant. The right eye, adapted to the brightness of the screen, views the test patch at E_1. A binocular match between two patches, seen simultaneously (see figure 4), is made, and the wedge reading W_1 is taken. The two patches in the right eye are kept equal during this experiment, so that $W_1 = W_2$. This procedure is carried out six times and a mean is taken. As the densities of

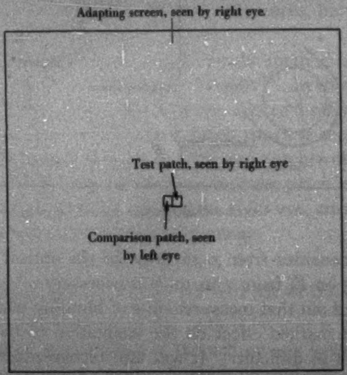
Adapting screen, seen by right eye
Test patch, seen by right eye
Comparison patch, seen by left eye

Figure 4. Schematic diagram of field of view.

the filters F_1 and F_2 are accurately known, the total density in the optical path is $F_1+F_2+W_1$, and from this the physical brightness of the test patch seen by the right eye can be found. Let this be B. The threshold value, or measurement of subjective black, for the same adapting-screen brightness, is measured by increasing the density of W_1 and W_2 in turn until no difference in brightness between the two halves of the bipartite field viewed at E_1 is experienced on still further increase of the density of either W_1 or W_2. The lowest density of W_1 (or W_2) when this state is reached, together with the densities of F_1 and F_2, gives a measure of the physical brightness at E_1 for the subjective black. A mean of six readings is taken; let this be B'. Thus the physical-brightness range between subjective black and the fixed apparent brightness as given by the comparison patch is $(B-B')$, where B and B' vary with the adaptation brightness to which the observer's eye is adapted. The number of just noticeable brightness-differences between B and B' is measured in the normal way, the right eye only being used. This eye, adapted to the necessary adapting-screen brightness which is viewed throughout the experiment, views the test field set at B'. Wedge W_1 is then altered to increase the intensity of one-half of the field until a just noticeable brightness-difference is seen between the two fields, the reading being recorded. This is repeated with wedge W_2, and so on, until the physical brightness B is reached. These readings give the number of just noticeable differences between B and B'. Adapting-screen brightnesses varying from 0·001 to 400 equivalent foot-candles were used. Curves are shown in figure 5.

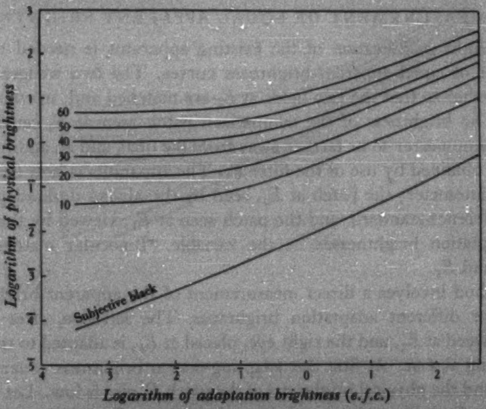

Figure 5. Number of just noticeable brightness differences from black.

The first readings were made with a fairly high value of B (about 10 e.f.c.) and it was found that the number of brightness differences between B and B' was the same for a large range of adaptation brightnesses. Such measurements, of course, cannot be made with precision, but the measurements at least indicated equality, the number of steps varying between 60 and 70. Further measurements showed, however, that when the value of B was lowered (to about 0·1 e.f.c.) the number of steps between B and B' was not independent of the adaptation. For instance, the first measurement for $(B-B')$, when B was high, gave the number of steps as varying from 60 to 70, but when the value of B was lowered so that at a low adaptation (almost dark-adaptation) the number of steps for the range B to B' was about eight, it was found that when the eye was adapted to a screen-brightness of about 400 e.f.c. the number of steps increased to about 50. Such a marked variation in the number of steps cannot be regarded as being due to experimental error, and it was decided to abandon, at least temporarily, the approach to tone-reproduction by the number-of-steps method and to treat the subject from a different viewpoint.

§ 4. MEASUREMENT OF EQUAL APPARENT BRIGHTNESS

Only a slight modification of the existing apparatus is needed to enable the measurement of equal-apparent-brightness curves. The two wedges W_1 and W_2 are used together so that the two fields at E_1 are matched and, in effect, act as one field only. The brightness of the comparison patch seen at E_2 can be varied by moving the lamp nearer to or farther away from the opal, and an additional range of brightness is obtained by use of the filter F_3. The apparatus is now used solely as a subjective photometer; the patch at E_2, seen by the always dark-adapted left eye, forms the reference standard, and the patch seen at E_1, viewed by the right eye at varying adaptation brightnesses, is the variable. Binocular matches are made between E_1 and E_2.

This method involves a direct measurement of the apparent brightness of the test patch for different adaptation brightness. The left eye, after being dark-adapted, is placed at E_2, and the right eye, placed at E_1, is adapted to the brightness of the adapting screen. At first this adapting-screen brightness is very low, about 0·001 e.f.c., and the physical brightness of the patch E_2 also is low. Let this physical brightness be A. A binocular match is made between E_1 and E_2 and the mean of three readings is taken. The physical brightness at E_2 is then raised, to B say, and again the mean of three binocular matches between E_1 and E_2 is taken. These measurements are repeated for physical brightnesses C, D, E, F and G at E_2. After these readings have been made, the adapting-screen brightness is raised to about 0·1 e.f.c. and the physical brightnesses of A, B, C, D, E, F, and G, exactly produced again at E_2, are again binocularly matched with the test patch at E_1. These measurements are then repeated with higher adapting-screen brightnesses up to 400 e.f.c. The physical brightness at E_1, in equivalent foot-candles, is found from the values of the wedge readings and filters F_1 and F_2. It should be remembered that the physical brightness of the comparison patch also fixes the apparent brightness, since the adaptation brightness of the left eye (dark-adaptation) is always constant.

During measurements it is necessary to keep the left eye as near as is possible to a state of dark-adaptation. A rubber cap fitted around the exit pupil E_2 prevents any extraneous light from entering the left eye, and in order that the light passing through E_2 may not adapt the left eye to any marked extent, this eye is opened only when actual measurements are being made. As the physical brightness of the patch of light seen at E_2 is never high, it is felt that the precautions adopted are sufficient to ensure that the left eye shall always remain in a state which is near enough to dark-adaptation for the particular experiments described.

If now the logarithms of all the physical brightnesses of the test patch corresponding to the physical brightness A of the comparison patch are plotted as ordinates against the logarithms of the corresponding adaptation brightness as abscissae, the line joining these points is a locus of equal apparent brightness over an adaptation-brightness range of 0·001 to 400 e.f.c. Values for the physical brightnesses B, C, D, E and F are similarly plotted, and the results are shown in figure 6. It should be realized that, when these measurements are made, the adapting field is always on and also acts as a surround field. The difference between the results of figures 5 and 6 is very striking and, as has been stated before, it was thought that the curves based on the number of just noticeable steps and the apparent brightness curves ought to have been of the same type. The difference was thought to be due to the influence of the surround field and so further experiments were made.

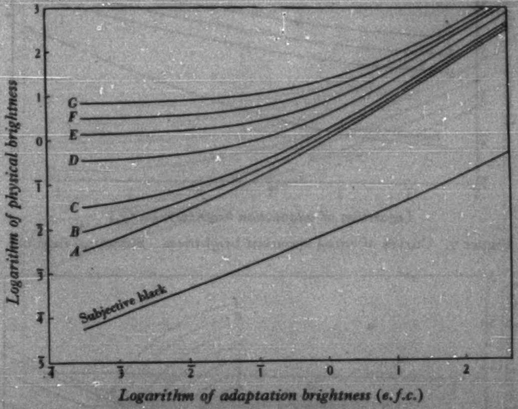

Figure 6. Curves of equal apparent brightness. Surround field on.

The experiments were very similar to those previously made except that, instead of the adapting field being on the whole time, the right eye was adapted to the adapting-screen brightness and then, just before the measurements were made, the adapting light was cut off. This ensured that the binocular matches should be made at the correct adaptation brightness, but the surround field was now black: such measurements may be regarded as being a function of adaptation brightness only.

The results are shown in figure 7, and it can be seen that the effect of the surround field is very considerable. It is obvious that, at low adaptation brightness (0·001 e.f.c.), the physical brightnesses of the test patch to match *A*, *B*, *C*, *D*, *E*, *F* and *G* must be the same whether the field is on or off, since in both cases the

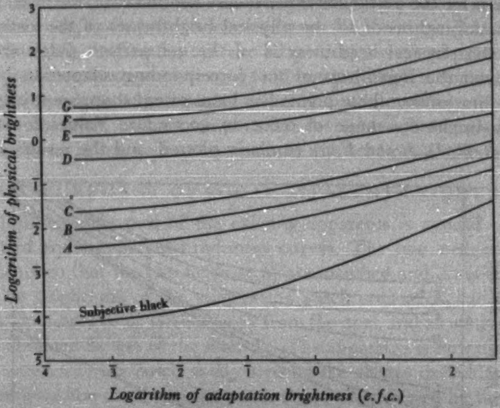

Figure 7. Curves of equal apparent brightness. Surround field off.

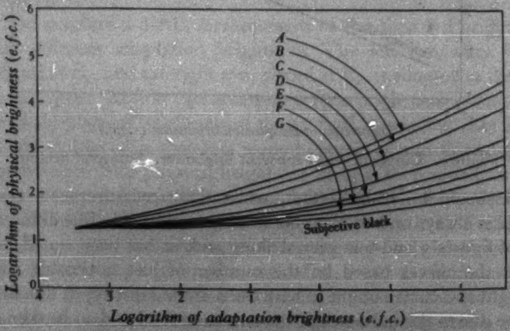

Figure 8. Differences in the logarithm of the physical brightness due to the surround field being on and being off.

surround field is approximately black, but as the adaptation brightness is increased the difference becomes steadily greater. The differences shown in figure 8, and the greatest difference occurs when the low physical brightness *A* and high adaptation brightness (400 e.f.c.) are used. Here a difference of 3·20 (0·7–$\bar{3}$·5) logarithmic units of brightness is caused by the surround field being left on. Thus, in order to maintain a match between the left eye and the right eye light patches when the adapting field is switched off, the physical brightness of the test patch seen by the right eye requires to be changed by a fraction varying between 1 and 1000, according to the magnitude of the physical brightness of the test patch and adaptation brightness. It is very important to note that, as far as can be measured with the existing apparatus, the effect of changing the surround field is instantaneous; the time of occurrence cannot be greater than about 0·2 sec., and is possibly considerably smaller. A great deal of importance may be attached to this effect, which may be regarded as being analogous to the effects described by Schouten(9). It is doubtful whether a photochemical process operating at the retina would act in such a short time, and the phenomenon appears to be almost certainly of the electrical type measured by Granit in Finland, and Graham, Hartline and others in America. This opinion is shared by Prof. R. Granit and Dr W. D. Wright, who have witnessed the effect in this laboratory. An explanation of the phenomenon would necessitate further experiments of the type made by the above-mentioned workers.

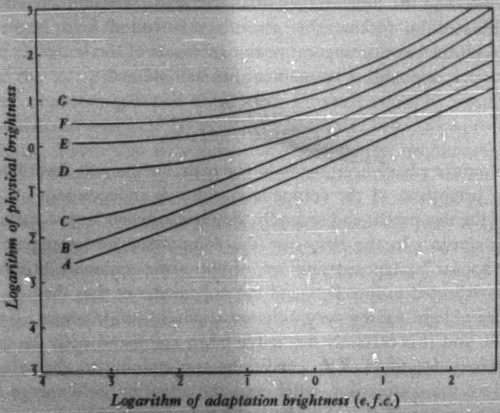

Figure 9. Curves as figure 5, but with 10° contrast field around test field.

It looks, prima facie, as if the subjective black follows some rule of its own. The results show that, when the eye is adapted to 400 e.f.c., the physical brightness of the subjective black is decreased by only eight times, as compared with the thousand-times decrease in the physical brightness *A*. It must be remembered, however, that the technique of measuring the black is different from the method of measuring the other brightnesses, the measurement being made by decreasing, in turn, the physical brightnesses of the two fields seen at E_1 by the right eye. Thus for the black the judgement is centred almost entirely on the two small test fields, and the contrast effect of the surround field is to some extent eliminated. This is believed to be the reason why the results of the number-of-steps method do not agree with the results of the apparent-brightness method, despite the fact that the adapting field is on while the measurements are being taken, figures 5 and 6. Results shown in figures 5 and 7 are in much closer agreement, and it is confidently expected that if the number of steps were measured with the surround field off, the agreement would be perfect within experimental error.

Further experiments were conducted to see whether the effect of the surround field could be wholly or partially eliminated, with the surround field on. For these, further slight modification of the apparatus was necessary. A small surround field subtending about 10° at E_1 was placed around the test patch. The brightness of this surround field varied with, and was always slightly darker than, the brightness of the test patch controlled by the wedges W_1 and W_2. The results, shown in figure 9, indicate that the effect of this small surround field does at least partially eliminate the contrast effect of the much larger surround field. Whether the effect could be entirely eliminated by making this secondary surround field larger cannot be determined with the existing apparatus, the apertures of the lenses L_1 and L_2 being only large enough to enable a 10° subtense to be obtained.

§ 5. DISCUSSION

Two assumptions have been made in the experiments just described: first, that the apparent brightness of the comparison patch is independent of the physical brightness of the test patch; and secondly, that it is independent of the brightness of the adapting screen. In the first case the comparison and test patches, viewed binocularly, appear juxtaposed and are of the same apparent brightness. They would not be expected to interfere, and the only evidence that they do so(10) refers to two patches of light having very different apparent brightnesses. Even then the effect is small, and it is certainly negligible when compared with the magnitude of the effect already described. The conditions appertaining to the second case are different. Here the comparison patch and the adapting field merge binocularly, and it might be expected that the physical brightness of the adapting field would affect the apparent brightness of the comparison patch. This effect was at one time thought to occur and to account for the fact that the number of brightness steps between two equal apparent brightnesses is different for different adaptation brightnesses. In respect of this point a passage may be quoted from Duke-Elder's *Text Book of Ophthalmology*(11). That author writes: "It was early remarked by Fechner(12) that a bright surface regarded with both eyes does not look brighter when looked at with one, a point which, as we have seen, the flicker experiments of Sherrington(13) demonstrated very clearly. When the stimulus presented to the two eyes is the same, therefore, there is no summation but the binocular sensation of brightness is not perceptibly different from either of the two uniocular components."

The experiments themselves also throw some light on the dependence of the apparent brightness of the comparison patch on the brightness of the adapting screen. When the physical brightness of the comparison patch is low (0·005 e.f.c.) and the brightness of the adapting screen is high (400·0 e.f.c.), no difficulty is experienced in distinguishing the comparison patch from the background. If the brightness of the adapting screen did seriously affect the apparent brightness of the comparison patch, it would be expected that this patch would disappear under the conditions just stated, since the contrast sensitivity at the high brightness (400 e.f.c.) is low. This viewpoint is further supported by the consideration that when the adapting field is switched off there is a considerable difference produced in the apparent brightness of the test field, but apparently no difference in the comparison spot.

A more stringent test was made to examine this point. A circle of black flock paper, about 2 in. diameter and having a reflection factor of 0·01, was placed close to the adapting screen in such a position that the part of the screen which appears to surround the comparison patch was substantially black. Under such conditions the brightness of the adapting screen cannot presumably influence the apparent brightness of the comparison patch. Measurements were then made on the apparent brightness of the test patch for different adaptation brightnesses. The differences between these measurements and those made formerly could be readily attributed to experimental error, and were certainly of a negligible order when compared with the main effects. From this it follows that the apparent brightness of the comparison patch is independent of the brightness of the adapting screen.

A further, independent, test was made by Dr W. D. Wright at the Imperial College of Science and Technology. His method was as follows. A small patch of light was viewed by the left eye, and a larger patch of considerably higher brightness was viewed by the right eye. The light entering the right eye could be switched on and off rapidly, and Dr Wright states that, by memory matching, he could discern no appreciable difference in the apparent brightness of the small patch of light whether the larger field was on or off.

In the light of the above results it seems reasonably certain, therefore, that the binocular addition of the adapting-screen image to the comparison field does not affect the apparent brightness of the latter to an extent sufficient to cause doubt on the reality of the contrast effect, although it is difficult to see why there should be this independence.

§ 6. CONCLUSIONS

The most important conclusions that may be drawn from this investigation is that there exists a contrast effect which is of very large magnitude and which acts almost instantaneously. Whilst it has not been specifically proved to be applicable in its full magnitude to the problem of photographic tone-reproduction, it may safely be assumed to be of some importance in that connexion.

It is well known that only part of the image falling on the fovea centralis of the eye is sharply focused; this applies to a field subtending approximately 2° at the eye. Consequently when a picture or scene is being viewed, only a very small part is seen in definite focus, and for a full appreciation of the picture or scene the eye has to move so that the whole may be focused on the fovea(2). The adaptation brightness will reach a definite and constant, or almost constant, amount for any one scene. The apparent brightness of any part of the scene is, however, dependent on the brightness of the immediately neighbouring parts. It may therefore be impossible, in consequence of this contrast effect alone, to construct sensation curves of the simple type postulated for consideration in the problem of tone-reproduction, for the reason that the apparent brightness of any surface is a function not only of the state of adaptation of the eye, but also of the brightness-distribution in the remainder of the scene.

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VIBRATIONS OF FREE SQUARE PLATES: PART 1. NORMAL VIBRATING MODES

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ABSTRACT. After a brief historical note, the paper describes the result of systematic observations which have been made on the normal vibrations of free square plates. The nodal systems are divided into seven classes and their further recognition is effected by the application of given rules. The natural vibration frequencies found by measurement are in fair agreement with Ritz's calculations. The relative frequencies of the higher partials agree with Chladni's observation that they are proportional to m^2+n^2. For values of m and n exceeding three, it is found that the frequencies, relatively to the gravest $1|1$ tone taken as unity, are given roughly by $\frac{3}{4}(m^2+n^2)$.

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§ 1. HISTORICAL NOTE

VIBRATING plates have now been a subject of study for one and a half centuries and this historical note is supplementary to the well-known accounts of the subject given by Rayleigh(14) and Love(15).

Chladni's early experiments on free square plates were described in his first book *Entdeckungen über die Theorie des Klanges*(1). The majority of his nodal figures are given again in *Die Akustik*(2). The largest and best arranged collection of figures can however only be seen in a later book, *Neue Beiträge zur Akustik*(3), which is now very rare, or else in Wheatstone's paper of the year 1833(8). In this important work, the whole of Chladni's set of normal nodal drawings precedes the great collection of Wheatstone's own constructed diagrams, and an account is also given of Chladni's final conclusions on the subject. It is interesting to find that Chladni's later notation, $\overline{m|n}$ and $\underline{m|n}$ (see § 3·1 below) is immediately comparable with the notation of the later approximate theory, and that his vibration frequencies, which extend as far as the $9|9$th mode, are expressed in numbers instead of in the notation of the chromatic scale.

Chladni's last paper was written in 1825(5), not, as stated by Melde(13), in criticism of foreign publications, but actually because of the curiously distorted figures and wrong conclusions of an early paper by Strehlke(4). Chladni died in 1827 and Strehlke published a second paper in 1830(6) in which his wrong conclusions are maintained. Faraday's(7) and Wheatstone's(8) papers appeared a few years later, and Strehlke's accurate drawings and measurements of some of the simpler nodal figures were first given in 1839⁽¹⁰⁾. It is interesting to read the publications of this period in chronological order. The criticism mentioned by Melde, or at any rate one that sounds very like it, is contained in Strehlke's last short paper which was published as late as 1872⁽¹²⁾.

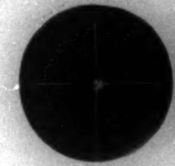
The approximate solution for the free square plate was given by Ritz⁽¹⁶⁾ in 1909. Ritz compared the results of his theory with the nodal figures of Chladni and Strehlke, and with Chladni's frequencies, given in the notation of the chromatic scale. The Chladni figures said to be missing are given in the *Neue Beiträge zur Akustik*. Since Ritz's work is not mentioned in Rayleigh's *Treatise*, it is well to draw attention, as has been done by Temple and Bickley⁽¹⁷⁾ and Southwell⁽¹⁸⁾, to Rayleigh's paper of 1911 entitled *On the calculation of Chladni's figures for a Square Plate*⁽¹⁹⁾. Commenting on Ritz's work, Rayleigh remarks, "The general method of approximation is very skilfully applied but I am surprised that Ritz should have regarded the method itself as new."

§ 2. SCOPE OF THE PRESENT OBSERVATIONS AND NOTES ON TECHNIQUE

It was evidently desirable that a systematic experimental study of the vibrations of free square plates should be undertaken with the resources of modern apparatus. This has already been done for the free circular plate⁽²³⁾, and the experimental details given in connexion with these need not be described again in detail. The vibrations were produced by the solid carbon dioxide method of excitation⁽²⁰, ²¹, ²²⁾ or else with a bow, and the same arrangements of support for the plates as in the previous work were generally employed. It is necessary to vary these according to the type of nodal figure which it is desired to produce, and for square plates an additional arrangement, in which the plates are suspended horizontally by means of two parallel loops of fine thread, was occasionally found convenient. The frequencies were measured in the usual manner by means of a calibrated valve oscillator. The observations were rendered independent of hygroscopic conditions by gently warming the plate and surrounding air by means of a bowl electric radiator, the heat being insufficient to alter either the elastic or damping properties of the material of the plate. A considerable number of *thin*⁽²⁴⁾ plates varying in size and material were employed. Details of some of these are given in table 3.

Most of the results described in the present paper are for brass plates, and a more detailed consideration of the dependence of both the actual and relative frequencies on Poisson's ratio is deferred to a later occasion.

It is well known that the variety of nodal figures which may be formed on square plates is very great, and one of the main purposes of the present work was to exhibit the results in a readily accessible form. To this end, the nodal figures of the normal vibrating modes have been arranged on one diagram, plate 2, so as to exhibit the natural groups into which they may be divided. The further identification of any nodal figure then becomes a very simple matter and can be effected by means of the rules given below. The natural relative vibration frequencies are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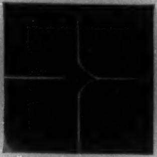
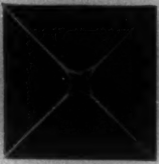

2 1|1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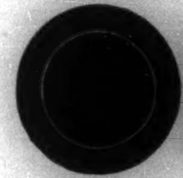
4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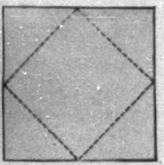
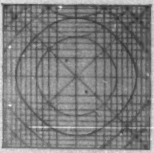
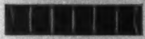
6 2|0+


7 2|0+

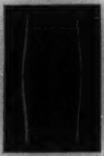

8 2|0+

9 2|0

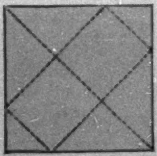

10 7|0

11 2|0

12 2|0


13 2|0

14 3|0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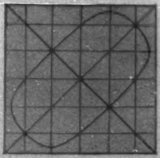
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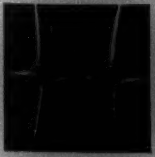
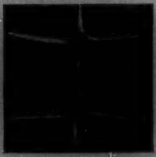

17 2|1


18 1|2

19 2|1−

20

See § 3. Figures 7 and 15 are hypothetical free membranes and figures 8 and 16 are vibrating free-bar constructions from Rayleigh. Figures 4 and 20 are superposed photographs.

shown in table 2, which corresponds in its arrangement with that of the diagram. Before describing the results and comparing them with those of other observers and with theory, we shall, however, discuss several preliminary matters and useful formulae. These are the subject of the several paragraphs of the next section.

§ 3. PRELIMINARY REMARKS

(3·1) *Considerations of symmetry and comparison with vibrating membrane* (see plate 1). In the first place it is interesting to follow Rayleigh(14) in tracing the continuity of the nodal figures as the form of the plate is gradually altered from a circle to a square. The indeterminate two nodal diameters of the circle, figure 1, plate 1, may give rise to one of the two nodal systems, figures 2 and 3, while the circle of figure 5 becomes the closed nodal system of figure 6. Systems similar to 2 and 3 may evidently be produced on any vibrating square surface. A closed system, comparable to 6, is also possible, although its exact shape and size will be determined by the nature of the restoring force (tension or elasticity) and by the prescribed boundary conditions (free, fixed or supported). Normal nodal figures, which are approximately those of a free square plate, may therefore be inferred from those of a hypothetical free membrane, for example from figure 7 or figure 15. Rayleigh has discussed the matter in some detail in his well-known chapters on membranes and plates(14). The appropriate equation for constructing the approximate nodal systems of a plate of side l is

$$\omega = \cos\frac{m\pi x}{l}\cos\frac{n\pi y}{l} \pm \cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} = 0 \qquad \text{......(1)},$$

where ω is the displacement of the plate at the point x, y, but the more general free-membrane relation

$$\omega = A\cos\frac{m\pi x}{l}\cos\frac{n\pi y}{l} \pm B\cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} = 0 \qquad \text{......(2)},$$

can be used when one of the numbers m, n is odd while the other is even (class 7 of table 1). In this case, the ratio of amplitudes $A \mid B$ of the superposed *single* vibrations may have any value. The physical meaning of the numbers m and n is made evident by the consideration that, when B is zero, the nodal system consists of m lines lying parallel to the y axis and n lines parallel to the x axis, the origin being taken at the lower left corner of the plate; see for example the $2 \mid 1$ system, figure 17. In other cases, as was so often emphasized by Chladni, there are, for fixed values of m and n, only two nodal figures, $\overline{m \mid n}$ and $\underline{m \mid n}$, the vibration frequencies of which are unequal. They correspond with the plus and minus signs in equation (1) and are referred to below as the $m \mid n+$ and $m \mid n-$ modes respectively. Tanaka(25), who gives equation (2), should not have applied it to the $7 \mid 3$ mode of a plate, and Colwell(26), who has constructed a number of nodal systems with the same formula, has also used it for the $5 \mid 3$ and $7 \mid 3$ modes where, m and n being both odd, it is not applicable.

The fact that equation (2) should not be used when the values of m and n are both even can be realized for the 2|0 mode, by considering the rectangular-plate figures, 9, 11–13. A narrow rectangle may vibrate like a bar with two straight nodal lines situated 0·224 of the length from either end. The manner in which these lines are modified when the rectangle is wider is shown in figures 9 and 11. Finally, when the rectangle becomes a square, the lines meet and form the diagonals of figure 3. Similarly, the figures of the wide rectangles 13 and 12 become, on a square plate, the closed nodal system of 6. This same matter has also received attention in Pavlík's(27) recent paper dealing with rectangular plates.

The reason why the above facts are sometimes overlooked is probably that it is so easy to obtain distorted figures unless the experimental conditions are perfect. Distorted examples of the 7|3 mode just mentioned are shown in plate 4, figure 3 and in plate 6, figure 8. These may be compared with plate 4, figure 2. There are at least four possible causes for such results, namely (1) a want of uniformity in the length of the sides or the thickness of the plate, (2) a directional difference in its elastic properties which is often caused by mechanical rolling, (3) a want of freedom in the method of support, (4) an element of forcing in the method of excitation; see also § 4·2.

(3·2) *Graphical construction of nodal systems by the superposition of vibrations of free bars.* It was early recognized that, just as the vibrations of square membranes could be derived from those of parallel wires vibrating at right angles, equation (2), so those of a square plate might, assuming Poisson's ratio to be zero, be constructed by the superposition of vibrating bars. Two of Rayleigh's(14) constructions are reproduced in figures 8 and 16 of plate 1, the method being that of Maxwell. It will be seen that they are much nearer to the actual photographs, figures 6 and 14, than are the membrane diagrams. They are in fact simple cases of the approximate constructions obtainable when two terms only of Ritz's series equation are retained, thus:

$$\omega = u_m(x)\, u_n(y) \pm u_n(x)\, u_m(y) = 0^{(16)} \qquad \ldots\ldots(3),$$

where the functions u are those proper to a free bar of length equal to the side of the plate. As before, when one of the values m, n, is odd while the other is even, additional figures are given by

$$\omega = Au_m(x)\, u_n(y) \pm Bu_n(x)\, u_m(y) = 0^{(16)} \qquad \ldots\ldots(4).$$

For greater accuracy it is necessary to retain a larger number of terms of Ritz's equation, as was done by Lemke(28) for the 2|0+ and 3|3 modes; her results are too lengthy to be quoted here.

An idea of how soon we may expect the simpler membrane construction from equation (1) to give reasonably accurate results for the nodal systems of upper partials, except near the edges, may be gained by looking at figure 10, which shows how nearly, in the 7|0 bar system, the nodal spacing approaches to the equal spacing of a vibrating wire. Attention is also drawn to the superposed prints of figures 20 and 4. The points of intersection of the nodal lines of the superposed vibrations, Strehlke's "poles", are always nodal$^{(8, 10)}$. This applies even in cases like 2|0, + or −, where, although the *single* vibrations are not possible, the poles are almost coincident with the nodes of free bars. Thus in figure 4 the poles are situated 0·228 of the length of side away from the edge, a distance which is only 2 per cent greater than for a bar, 0·224. These two numbers would presumably coincide if the value of Poisson's ratio were zero. The measurements are found to be in excellent agreement with those given by Strehlke$^{(9)}$ at the end of his paper on vibrating bars.

(3·3) *Calculation of the vibration frequencies.* The natural frequencies f are given by

$$\lambda = \frac{192\pi^2 f^2 (1-\sigma^2)\rho}{Et^2}^{(16)} \qquad \text{......(5)}$$

for a plate of thickness t and length unity. E is Young's modulus, σ Poisson's ratio, and ρ the density. The values of the constant λ have been calculated by Ritz to the 6|6 mode for $\sigma=0{\cdot}225$ and by Lemke$^{(28)}$ for some other values of σ. It follows that for a plate of side l, the fundamental frequency

$$f = 0{\cdot}6577 \text{ (or } 0{\cdot}6277)\, c\frac{t}{l^2}; \quad \sigma = \tfrac{1}{4} \text{ (or } \tfrac{1}{3}) \qquad \text{......(6)}$$

where c is the velocity of sound appropriate to the material. Thus the given increase in Poisson's ratio produces a decrease of frequency of 4·8 per cent. This result is convenient for calculations in connexion with materials such as glass or steel ($\sigma \doteqdot \frac{1}{4}$) as compared with brass ($\sigma \doteqdot \frac{1}{3}$). It is also to be noted that since the variation of λ with σ is approximately linear$^{(28)}$, frequencies corresponding to other values of Poisson's ratio are easy to estimate with reasonable accuracy. When, for example, σ is $\frac{1}{5}$, the frequency should be equal to about $\frac{2}{3}.c.t/l^2$.

Relative frequencies. Ritz states that the natural frequencies are roughly proportional to

$$\sqrt{\{m^4+n^4+2(1-\sigma)\,m^2n^2\}} \qquad \text{......(7).}$$

This relation is evidently meant for higher partials where the distinction between plus and minus systems can be neglected. It is seen also that according to equation (7), when m and n are equal, the relative frequencies are unaffected by the value of Poisson's ratio. In view of these facts it is simpler to use Chladni's expression

$$m^2+n^2 \qquad \text{......(8)}$$

for obtaining the frequencies of upper partials in relation to one another.

According to the present observations (table 4) their values relatively to the gravest 1|1 tone taken as unity are given approximately by

$$\tfrac{3}{4}(m^2+n^2) \qquad \text{......(9)}$$

for values of m and n exceeding 3.

We may also note that when either m or n is zero, the sequence of tones according to Chladni is nearly that of a free bar, $(3{\cdot}011)^2, 5^2, 7^2, \ldots$. This relation seems to fit best when the mean value of the 2|0+ and 2|0− frequencies is used for the first member of the series.

§ 4. EXPERIMENTAL RESULTS: I. THE NODAL SYSTEMS

(4·1) *The seven classes of nodal symmetry.* In plate 2 the normal nodal systems are arranged on a square diagram which exhibits the seven classes into which they may be divided according to the nodal design at or near the centre. The systems for which the values of m and n are equal occupy the squares which extend diagonally from the top left corner to the bottom right corner of the page, the $6|6$ space remaining unfilled. The $m|n-$ systems occupy the triangular space to the right of this diagonal, and the $m|n+$ systems the triangular space to the left of it. A general study of alternate figures and of the development which occurs in passing along the rows, columns or diagonals, results in the following classification.

Table 1. The seven classes of nodal symmetry of free square plates, see plate 2

e denotes that the values of m and n are even, and o that they are odd

Specification of vibrating mode	Condition at centre	Further details of nodal lines at or near centre	Rotational symmetry	Class abbreviation symbol	Further examples see plates
$m=n$					
1 $o\|o$	Node	Parallel to edges	90°	+	
2 $e\|e$	Antinode	Parallel to edges	90°	□	
$m\neq n$					
3 $o\|o-$	Node	Diagonals and medians	90°	✱	4
4 $e\|e-$	Node	Diagonals	90°	×	5
5 $o\|o+$	Node	Medians	90°	+	6
6 $e\|e+$	Antinode	Closed figure	90°	○	6
7 $o\|e+$ or $-$	Node	One diagonal	180°	╱ or ╲	7, 8

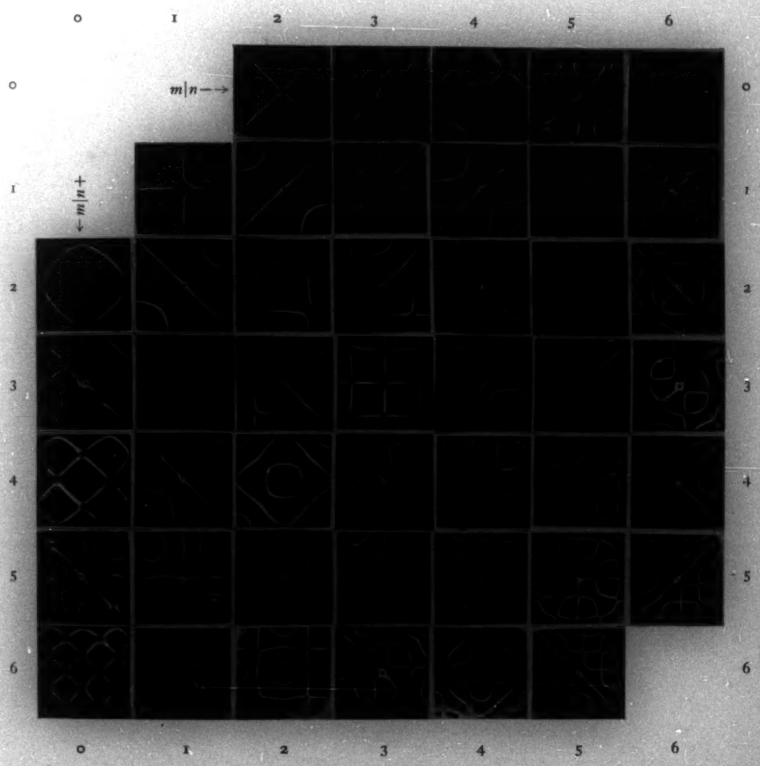
Equations (2) and (4) can be used for class 7 only.

The division into seven classes, arrived at by inspection of the nodal system, is of course consistent with Ritz's analysis. Ritz distinguishes only five classes, the first and second of table 1 being included in the "plus" fifth and sixth classes respectively. Since however classes 1 and 2 have distinctive nodal systems and occupy a peculiar position in plate 2 it is convenient to think of them separately. It will be noted that there are no corresponding special cases for the "minus" classes 3 or 4; analytically the plate remains at rest, when m and n are equal, and in plate 2 all the spaces are already accounted for.

(4·2) *Recognition of the nodal systems. Higher overtones.* The recognition of a normal nodal system can be effected with certainty and ease if it is first of all placed in one of the classes of table 1. We then already know whether the values of m and n are odd or even, and it only remains to determine their actual values.

When m and n are equal, classes 1 and 2, recognition is immediate since the nodal lines, numbering m, run approximately parallel to the sides.

In the remaining classes one of the numbers m, n (say m) is counted either across the plate or more generally along one edge. The manner of finding n then varies according to the system, but, except in class 3, which will be considered last, n also can be determined in a few moments. Thus, in classes 5, 6 and 7, the



Normal nodal systems of free square plate, see equations (1) and (3). The nodal designs are distinct for given values of m and n except when one of the values of m, n is even when the other one is odd. See table 1.

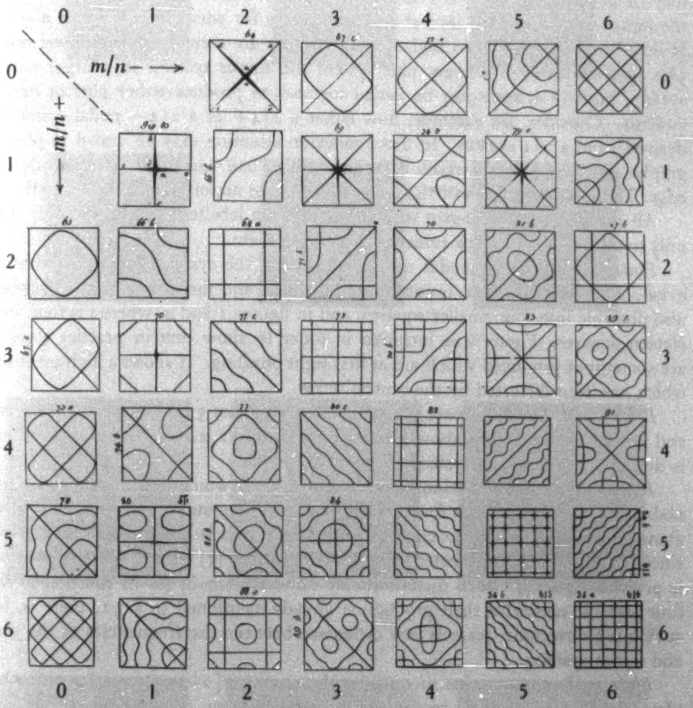


Plate 3. Selection of Chladni's drawings arranged as in plate 2, see § 5.

number of nodal lines which cut a diameter which is not nodal always equals $m+n$, a relation which can be verified on plate 2 and also on plates 6–8.

The nodal systems of classes 4 and 6 are divisible (if the edge effects are neglected) into four smaller squares for which m and n have half their former values; see plates 2, 5 and 6, and the examples given by Chladni(2). His observation may be generalized, for it is evident that when m and n have a common factor s the figure can be divided into s^2 smaller squares for which $m'=m/s$ and $n'=n/s$. It may be noted that when higher nodal designs are composed from lower ones, plus systems result in a larger plus system and minus systems in a larger minus system, but $o|e$ systems can be joined together to produce either plus or minus systems. Consider, for example, how either a $2|4+$ or a $2|4-$ nodal system is derived from a $2|1$ system. In this connexion attention may be drawn to photographs 1 to 4 of plate 7, which have been cut in half and fitted together edge to edge. These photographs will also give an idea of the importance of the edge effects.

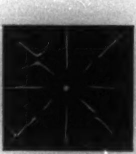
All the photographs shown in plates 4 to 8 are labelled, and accordingly it is only necessary to add a few remarks concerning them.

Plate 6 (1). The antinodal centres show that the systems belong to class 6, $e|e+$. They are identified immediately by the m and the $m+n$ count. They are also divisible into four smaller squares, and in figures 1 and 4, where s is four, into sixteen squares. Figure 6 is included in order to show that in practice designs are sometimes produced which are at first sight puzzling. It shows a high overtone which has been distorted by the central screw.

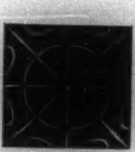
Plate 6 (2). The figures are placed in the $o|o+$ class 5 by looking at the centre, and are then identified as above. In figure 10, since s is three, the $9|3+$ system is divisible into nine $3|1$ systems.

Plate 5. The systems belong to class 4, $e|e-$, for which both the diagonals are nodal, but by division into four smaller squares the systems can be identified as above unless they belong to class 3. In figure 2, $8|2$ is not theoretically perfect since the amplitude A is greater than B; see equations (1) to (4). A better example is given in figure 4. Such distortions are not uncommon, for in addition to the four possible causes for their production already mentioned in § 3·1, there is, for such a high overtone, scarcely any difference between the frequencies of the plus and minus systems.

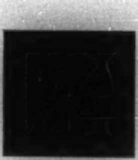
Plate 4. It now remains to consider the class 3, $o|o-$ systems shown in this plate, which cannot all be recognized by means of one comprehensive rule and need, therefore, to be looked at in more detail than the figures of the other classes. The value of m is found as before. All the systems for which $m-n=2$ present the typical appearance seen already in plate 2, and now in figures 6, 7 and 10, and $m+n$ can be determined by counting from one corner to the opposite corner along two adjacent sides. Another way of stating this rule is that m is given by counting along one side including the diameters, and n is obtained by counting along the next side excluding the diameters. In applying this relation to the systems for which $m-n=4$, see plate 2, $6|2-$, and figures 2 and 5, the *loops* count as two along the first side and as one along the second side. This result may be compared



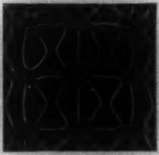
1 7|1−



2 7|3−



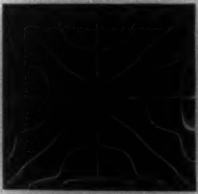
3 7|3−



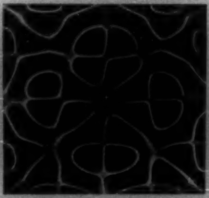
4 9|3−

5 9|5−

6 9|7−



7 7|5−

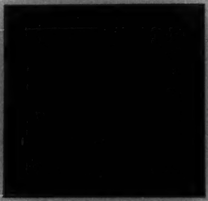


8 9|5−

9 11|3−, i

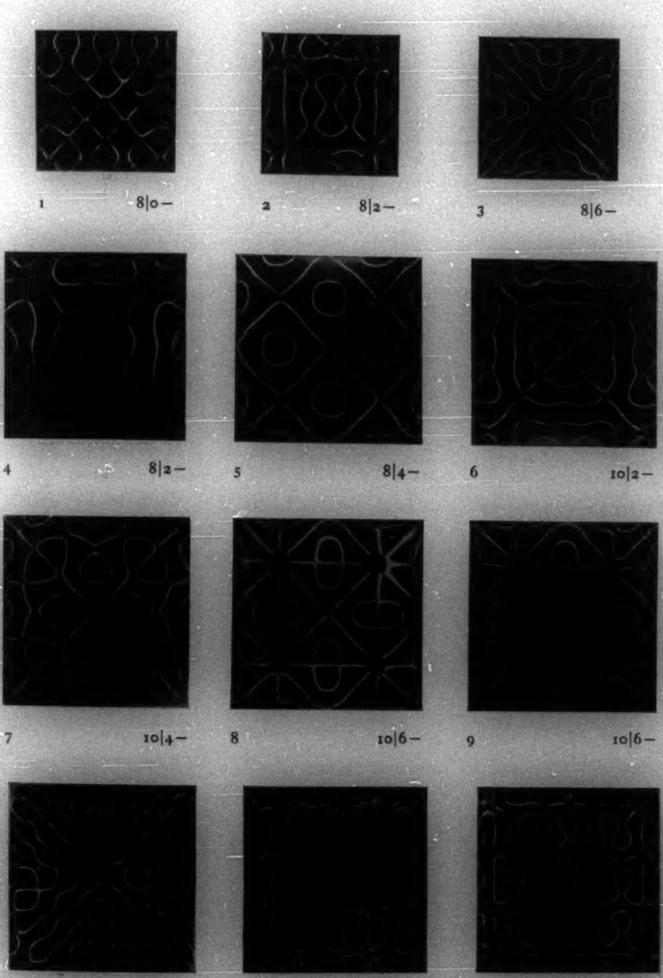
10 11|9−

11 13|3−



12 13|7−

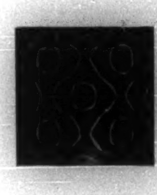
Nodal systems of *o*|*o*−, class 3, see table 1 and § 4·2.



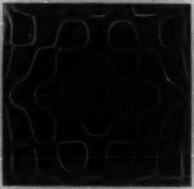
1 8|0−
2 8|2−
3 8|6−
4 8|2−
5 8|4−
6 10|2−
7 10|4−
8 10|6−
9 10|6−
10 10|8−
11 14|4−
12 14|4

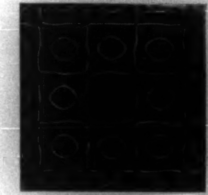
Nodal systems of $e|e-$, class 4, see table 1 and § 4·2.

(1)


1 8|4+

2 8|6+

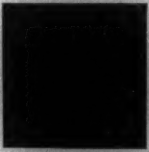

3 8|6+


4 12|4+

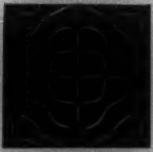
5 12|10+

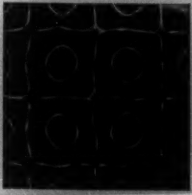
6 10|8+

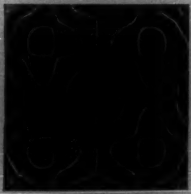
(2)


7 9|1+

8 7|3+


9 7|5+

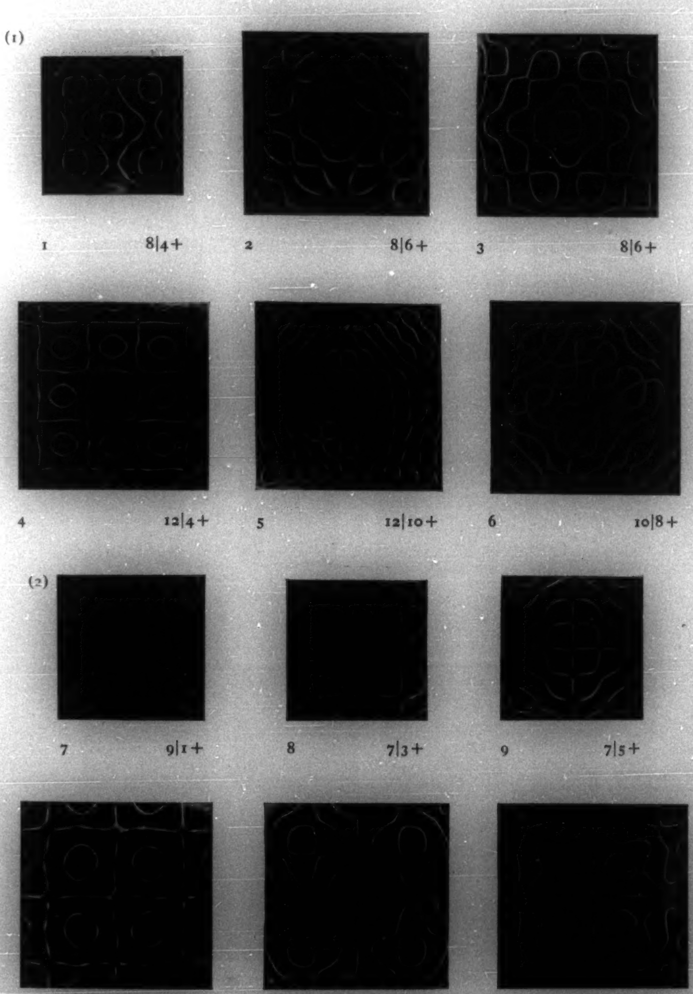

10 9|3+


11 9|5+

12 11|3+

Nodal systems of (1) $e|e+$, class 6, (2) $o|o+$, class 5, see table 1 and § 4·2.

To face page 838



(1)

1 8|4+ 2 8|6+ 3 8|6+

4 12|4+ 5 12|10+ 6 10|8+

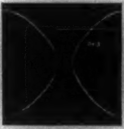
(2)

7 9|1+ 8 7|3+ 9 7|5+

10 9|3+ 11 9|5+ 12 11|3+

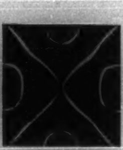
Nodal systems of (1) $e|e+$, class 6, (2) $o|o+$, class 5, see table 1 and § 4·2.

(1)


1

2

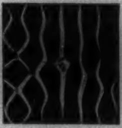
3


4

(2)

5 3|0

6 5|0

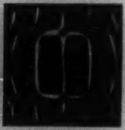

7 7|0

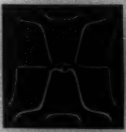
8 4|1

9 4|1

10 8|1

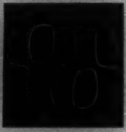
11 3|2


12 5|2


13 4|3

14 6|3

15 6|3


16 6|3

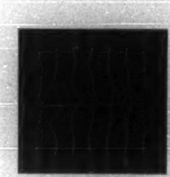
17 6|1

18 6|0−

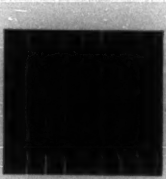
19 7|4

20 13|0

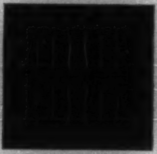
(1) Showing edge effect, see § 4·2. (2) Nodal systems of $o|e$, class 7 (except figure 18), see table 1 and § 4·3



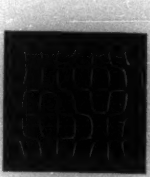
1 9|0



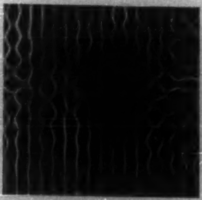
2 9|2



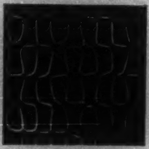
3 10|3



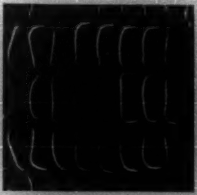
4 9|6



5 17|0

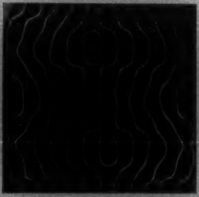


6 10|5

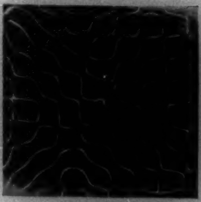


7 9|4

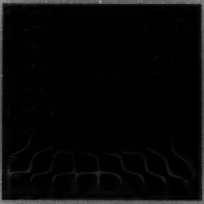
8 9|4



9 13|0



10 9|8



11 10|9

12 12|3

Nodal systems of *o*|*e*, class 7, see table 1 and § 4·3

with somewhat similar conclusions in connexion with the compounded modes of circular plates$^{(23b)}$. A second distorted version of 9|5− is given in figure 8 in order that the three-line prongs which are frequently met with in practice may be made clear. It is interesting to find a nodal circle in the 7|3− figure; it is the only example so far obtained. Another point occurs in connexion with the systems for which $m-n=6$, see 7|1−, figure 1, and the slightly distorted 13|7−, figure 12. After m has been counted as usual along one of the sides it is seen that the lines at right angles to diameters are not included again in the n count. The 9|3− mode incidentally, where $s=3$, is divisible into nine 3|1 systems. The lattice appearance of the figure may be noticed, since it is sometimes stated that nodal lines approximately parallel to the sides only occur when m and n are equal. Further examples may be found in other plates. The lattice is sometimes of considerable assistance in identifying the class-3 systems, as for example in figures 9 and 11.

(4·3) *Further observations on the o|e nodal systems, plates 7 and 8.* As has been mentioned in § 3, when one of the numbers m, n is odd while the other is even, equations (2) or (4) are applicable. There is no distinction (except a formal one of rotation through a right angle, see plate 2) between the plus and the minus nodal systems, and there is no limit to the variety of nodal figures which may be obtained for fixed values of m and n. *Single* vibrations, corresponding to zero values of the second term of equations (2) or (4), are shown in some of the photographs of plates 7 and 8. The 3|0, figure 5, is similar to one of Chladni's, and one and a half *Biegungen* or bendings are visible in the nodal lines. Had Chladni multiplied all his data regarding these bendings by two it is possible that he would have recognized their significance. In the present case the three half-bends indicate that a trace of the 0|3 system is present. This matter has been fully discussed in the paper on free circular plates$^{(23b)}$. One $e|e$ distorted nodal system is included in plate 7, figure 18. It will be noted that were the bendings slightly more pronounced we should obtain the normal 6|0− mode of plate 2. The closed nodal lines of 7|4, figure 19, should join, as can be seen from the diagonal count; cf. Chladni's version$^{(3)}$. Attention may be drawn to the characteristic single nodal line through the centre and the 180° rotational symmetry which persists in all the figures of plate 8. Broken patterns, such as those shown in plate 7, figure 20 and plate 8, figure 5, are quite commonly produced on large plates which are difficult to obtain uniform.

§ 5. COMPARISON OF RESULTS: I. THE NODAL SYSTEMS

A selection of Chladni's drawings, arranged as in plate 2, are shown in plate 3. Most of them are from *Die Akustik*, those with no numbering are taken from Tyndall's *Sound*, while the 5|1+, 6|5 and 6|6 gaps have lately been filled from the collection given in *Neue Beiträge zur Akustik*.

The nodal systems of plates 2 and 3 may be conveniently compared along the diagonal series of figures for which the values of m and n are equal and thereafter on either side of this. The remarks which follow also include a comparison where necessary with the diagrams given in Ritz's paper.

$m=n$. Except for 1|1, the actual figures vary considerably from the idealized lattice drawings given by Chladni. Strehlke's version (given by Ritz) of the 2|2 system is essentially more correct, though it tends to over-emphasize the curves of the four nodal lines. Lemke's(28) elaborate construction of the 3|3 system is practically identical with the present photograph, and shows that when sufficient terms of the Ritz equation are retained, the theory is in good agreement with practice. The 5|5 photograph is of special interest and will, it is hoped, be referred to again in a later communication on the subject of compounded modes: a trace of a second 7|1 tone, of near period, can be detected in it. Chladni(3) gives a suggestive distortion of the 6|6 lattice shown in plate 3, which consists of 12 parallel wavy lines similar to the 11 wavy lines of the correct 6|5 figure.

$m-n=1$. In Chladni's version of the 2|1 mode A and B are unequal; the more typical figure is shown in the photograph. The peaks of the wavy lines in the photographs are more pointed than those given by Chladni or by Ritz.

$m-n=2$. This familiar series of nodal systems is very easily produced. Examples as far as 11|9− may be found on plates 2, 4 or 5, and as far as 12|10+ (except 9|7+) on plates 2 and 6.

$m-n=3$. The 3|0 system shown in the photograph has been produced many times. I have never obtained the figure as it is given by Chladni and Strehlke. Other differences in detail may be noted in this series.

$m-n=4$. The 4|0 photograph differs considerably from the versions given by Chladni and Ritz. The two nodal diameters which are missing in Chladni's 6|2− drawing are included in his later book(3). None of the drawings really convey the character of this familiar figure. The 5|1 mode tends to combine with the 5|0 system as shown in the photograph and in Chladni's later drawings(3).

§ 6. EXPERIMENTAL RESULTS: II. THE NATURAL FREQUENCIES

The natural frequencies, relative to that of the 1|1 tone taken as unity, are given in table 2, which corresponds in its arrangement with the nodal systems of plate 2. The table extends over an interval of about 140, that is to say, over more than seven octaves. The higher figures are included so as to be available in a subsequent study of compounded normal modes. The difference between the frequencies of the "plus" and "minus" modes, which are printed in black type, are to be noted, and also the manner in which this difference rapidly decreases with the higher tones. The frequencies of the plus modes should not be greater than those of the minus modes were Poisson's ratio equal to zero.

The actual 1|1 frequencies for some of the plates employed, together with other particulars, are given in table 3. The observed frequencies of the fourth column may be compared with those, calculated by means of equation (6), which are given in the last column. Considering that, except for brass, the values for c and σ are assumed, the agreement between the two columns is very satisfactory.

For brass, c was measured from figures 1 or 5, plate 8, by means of the relation used by A. B. Wood and F. D. Smith[29],

$$c=\frac{2\sqrt{3}}{\pi}\frac{\lambda}{t}c_t \qquad \text{......(10)}$$

in which c_t is the velocity, and λ the wave-length of the transverse wave. The result, $35 \cdot 6 \times 10^4$ cm./sec. for the particular brass used, is in near agreement with the value given by Kaye and Laby, $36 \cdot 5 \times 10^4$.

Table 2. Relative frequencies of normal vibrating modes of free square brass plate

$m \mid n-$ $m \mid n+$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
0	—	—	**1·52**	5·10	**9·14**	15·8	**23·0**	32·5	43	55·2	70	*84*	*101*	*119*	141	0
1	—	1*	2·71	**5·30**	10·3	**15·8**	23·9	**32·2**	43	55·8	71	86·1	102	*121*	(*143*)	1
2	**1·94**	2·71	4·81	8·52	**12·4**	19·0	**26·4**	34	46·6	59	73	89	105	*124*	(*146*)	2
3	5·10	**6·00**	8·52	11·8	16·6	**22·6**	30·0	**39·5**	50·5	63·4	77·5	92·4	110	128	(*151*)	3
4	**9·9**	10·3	**13·2**	16·6	21·5	28·7	**35·5**	45·4	55·9	69·7	82·9	99	116	*132*	(*155*)	4
5	15·8	**16·6**	19·0	**23·3**	28·7	35	43	**52·1**	64·5	75·9	90	106	*122*	136	(*161*)	5
6	**23·8**	23·9	**27·1**	30·0	**35·9**	43	*51*	61·7	*73*	84	99	115	*130*	(*147*)	—	6
7	32·5	**32·4**	34	**39·8**	45·4	**53**	61·7	*70·3*	84	93	*108*	*124*	(*140*)	(*161*)	—	7
8	43·0	43·0	46·6	50·5	57·2	64·5	*73*	84	*94·4*	106	120	*136*	(*153*)	(*175*)	—	8
9	*55·2*	55·8	59	63·4	69·7	76·2	84	93·2	106	*120*	*133*	(*150*)	(*168*)	—	—	9
10	70	71	73	77·5	82·9	90	99	*108*	120	*133*	(*149*)	(*165*)	—	—	—	10
11	*84*	86·1	89	92·4	99	106	115	*124*	136	(*150*)	(*165*)	(*178*)	—	—	—	11
12	101	102	105	110	116	*122*	*130*	(*140*)	(*153*)	(*168*)	(*185*)	—	—	—	—	12
13	*119*	*121*	*124*	128	132	136	*147*	(*161*)	(*175*)	—	—	—	—	—	—	13
14	141	(*143*)	(*146*)	(*151*)	(*155*)	(*161*)	—	—	—	—	—	—	—	—	—	14
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	

* See table 3 for actual fundamental frequencies.

The numbers in italics are approximate only, and those in brackets have been determined by extrapolation. Any circular arc whose centre is at the top left corner will pass through nearly equal frequencies.

§7. COMPARISON OF RESULTS: II. THE NATURAL FREQUENCIES

It has already been shown by the numbers given in the fourth and last columns of table 3 that the experimental values of the fundamental frequencies agree well with those obtained by calculation. With regard to overtones, in order to avoid lengthy tables it has been considered sufficient to restrict the comparison of results to the series of partials for which m and n are equal, as shown in table 4.

Table 3. Fundamental 1|1 frequencies

Material of plate	Dimensions of plate (cm.)		Observed frequency (c./sec.)	Velocity c of sound (cm./sec. $\times 10^4$)	Poisson's ratio, σ	Calculated frequency*
	Length of side	Thickness				
Brass	6·00	0·201	1222	35·6	$\frac{1}{3}$	1248
	10·16	0·2035	432	,,	,,	442
	14·89	0·2135	206·5	,,	,,	208
	20·28	0·488	260	,,	,,	264
	19·89	0·1848	104	,,	,,	104
	20·44	0·164	87·7	,,	,,	87
	30·6	0·197	†46	,,	,,	44
	40·71	0·204	†27	,,	,,	25
Glass	20·4	0·3081	256	50	$\frac{1}{4}$	243
Steel	20·38	0·1935	157	50	$\frac{1}{4}$	153·2
Aluminium	20·49	0·2016	158	51	$\frac{1}{3}$	153·8

* By equation (6). † Inferred from second or third tone.

The values for c and σ are from Kaye and Laby's tables except that, for brass, c was determined from figures 1 or 5, plate 8(29).

Table 4. Comparison of results. Relative frequencies

$m \mid n$	1|1	2|2	3|3	4|4	5|5	6|6	7|7	8|8	9|9	10|10
Chladni, observed, glass*	1	4·58	10·8	18·5	30	42·7	60	80	102	—
Ritz, calculated, $\sigma = 0{\cdot}225$†	1	4·63	11·2	21·0	33·2	48·1	—	—	—	—
Pavlík, observed, stainless steel	1	4·34	9·33	—	—	—	—	—	—	—
Brass plates										
Lemke, calculated‡	1	4·87	11·8	—	—	—	—	—	—	—
observed	1	4·81	11·8	—	—	—	—	—	—	—
Waller, observed	1	4·81	11·8	21·5	35	51	70·3	94·4	120	149
$\frac{3}{4}(m^2+n^2)$§	—	—	—	24	37	54	73	96	121	151

* The numbers given in *Neue Beiträge zur Akustik* have been divided by six.
† From Ritz's values of λ, see equation (5). ‡ From Lemke's actual frequencies.
§ A convenient approximate formula for higher overtones.

Since frequencies of the lower tones are somewhat dependent on the value of Poisson's ratio, it is not to be expected that the early numbers given in the first three rows will be the same as those given later for plates made of brass. Pavlík's(27) numbers appear to be too low. This may be due to his experiments having been made on small plates in which the length of side, about 1 cm., was only ten times greater than the thickness; such plates can scarcely be regarded as thin(24).

The relative frequencies of the higher tones as given by Chladni are seen to be considerably less than those which are required by the theory or obtained in the present observations. It must have been difficult to determine the higher frequencies accurately with the means at Chladni's disposal.

The approximate agreement in the numbers of the last two rows is to be noted. This indicates that a rough estimation of the frequencies of the higher tones can be made using the formula (9) suggested earlier in this paper.

§8. CONCLUSIONS

1. Chladni's maturest work on the subject of vibrating square plates are given only in his last book *Neue Beiträge zur Akustik*. His complete set of normal nodal drawings and observed natural frequencies are reproduced by Wheatstone⁽⁸⁾ (§ 1).

2. The Ritz approximate method of solution of the problem of the vibrating free square plate is an extension of the Rayleigh method (§ 1).

3. The normal nodal systems may be arranged in a systematic manner on one diagram, plate 2, which shows that there are seven classes of nodal symmetry. These may be recalled by means of descriptive abbreviation symbols, see table 1.

4. Any nodal figure after being placed in its class, may be identified by means of simple rules (§ 4·2). For example, except when both diameters are nodal, the value $m+n$ is obtained by counting the number of lines across a diameter, and the value of m by counting along an edge or across the plate.

When m and n have a common factor s, the nodal system is divisible into s^2 smaller systems for which $m'=m/s$ and $n'=n/s$.

5. There are a variety of causes, see § 3·1 and § 4·3, for the distorted nodal figures which are in practice so commonly produced.

6. When the figures are being constructed graphically, it is important to recall that it is only when one of the values of m, n is odd while the other is even that an indefinitely large number of nodal figures may be constructed with fixed values of m and n (§ 3·1).

7. Since it is impossible for more than four nodal lines to pass through the centre of the square plate, the vibration amplitude for any mode is usually sufficient to move the sand over most of the surface. This result may be contrasted with the circular plate for which, when the number of nodal diameters are numerous, the sand remains quiescent except near the edge⁽²³⁾. The subject is of interest in connexion with the radiation of sound.

There is great variation both in the persistence of vibration and in the loudness of the various overtones.

8. The natural frequencies found by measurement are in fair agreement with Ritz's calculations, see § 3·3 and tables 3 and 4. The frequency of the gravest tone, when Poisson's ratio is $\frac{1}{3}$, is given approximately by $\frac{2}{3}\,ct/l^2$ for a plate of thickness t and length of side l where c is the velocity of sound appropriate to the material. The frequency decreases slowly and approximately linearly with increasing value of Poisson's ratio, see § 3·3.

9. It is found experimentally that the frequencies of the higher partials relatively to the gravest tone taken as unity are given roughly by $\frac{3}{4}(m^2+n^2)$, see table 4. When the frequencies are arranged as in table 2, any circular arc with centre at the top left corner passes close to numbers which are very equal.

§ 9. ACKNOWLEDGEMENTS

I am glad to express my cordial thanks to Professor Andrade for the loan of his rare copy of the *Neue Beiträge zur Akustik*; to Mr Ralph Cory, the Librarian at the Royal Institution, for enabling me to obtain the photographs from *Die Akustik*; to Imperial Chemical Industries Ltd. for giving the solid carbon dioxide, Drikold, used to excite the plates; and to the Trustees of the Augustus and Alice Waller Memorial Research Fund for providing some of the apparatus used during the investigations.

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FURTHER RESULTS ON THE MAGNETISM OF CHLORIDES OF THE PALLADIUM AND PLATINUM TRIADS OF ELEMENTS

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ABSTRACT. Measurements have been made on the variation of the magnetic susceptibility of the salts $RhCl_3$, $OsCl_2$, $IrCl_3$, $PtCl_2$ with temperature over a range of about 300° from room-temperature upwards. A torsion balance with electromagnetic compensation, of greater sensitivity than those previously described, and a special electromagnet were used. The results for the corresponding cations may be represented by an equation of the type $(\chi+K)(T+\Delta)=C$.

§ 1. SURVEY OF PREVIOUS WORK

The electronic group responsible for the magnetic moment of an atom, which we may call the "magnetogenic group", is symbolized by $3d$ for the iron family, $4d$ for the palladium family, $5d$ for the platinum family, and $4f$ for the rare earths. In the first three families the magnetogenic group is that of highest total quantum number and may be said to form the surface of the cations, but in the fourth this group is below the surface. A comparison of the experimental magnetic moments with those calculated for free ions shows that the magnetogenic group is practically uninfluenced by the molecular constitution in the rare-earth family, but in the other three families this influence is very marked, the effect being greater the greater the quantum number n of the group.

This fact is closely connected with the well-known alteration in the valency characteristics. Werner and Pfeiffer have pointed out that the halides of the triads (Fe, Co, Ni), (Ru, Rh, Pd), and (Os, Ir, Pt) deviate more and more from the alkaline-halide type in this order. For the first triad we can still speak of the electrolytical dissociation of the salts, whereas for the second, and more especially for the third, the natural salinity of the compounds is a debatable point.

In this paper we shall deal with further investigations of a series of chlorides of these families, specially prepared by C. W. Hereaus, of Hanau, which we first studied some years ago(1). At that time we measured the susceptibilities at room-temperature by the Faraday method, using a torsion balance and an electromagnet transportable at right angles to the axis of the magnetic field. The measurement made was that of the difference between the maximum angular displacements of the balance on either side as the magnet was moved. The value of χ_c, the susceptibility of the cation, was obtained relative to water.

By this method we obtained the values of the susceptibility at 20° C. set down in the following table, which includes also values obtained more recently by Cabrera and Fahlenbrach⁽²⁾.

Table 1. Experimental values for the gramionic susceptibilities of the cations of the chlorides of the palladium and platinum triads. (Unit $\times 10^{-6}$)

Cation	Cabrera and Duperier	Cabrera and Fahlenbrach
Ru^{+++}	1912·0	—
Rh^{+++}	45·5	48·4
Pd^{++}	3·38	2·23
Os^{++}	82·75	—
Ir^{+++}	47·2	—
Pt^{++}	17·4	—

Cabrera and Fahlenbrach's results provide a satisfactory confirmation of Cabrera and Duperier's, and for this reason we adopt the Cabrera-Duperier values for the purposes of the present investigation.

The variation of χ_c with temperature was also measured by the Faraday method, but a different balance, with electromagnetic compensation, was utilized. The sensitivity was less than that of the first balance, and in our measurements it was used to the limits of its sensitivity. For this reason a repetition of the study seemed desirable.

Cabrera and Fahlenbrach have since carried out more accurate temperature variation measurements with Ru^{+++}, Rh^{+++} and Pd^{++}. For the first cation the result differs but little from that obtained previously, and may be expressed by the equation

$$(\chi+K)\,T=C_{Ru^{+++}},$$

where $K=2{\cdot}16\times 10^{-4}$. This value of K indicates a diamagnetism additional to the paramagnetism, but its numerical value is greater than those usually found for this class of substances. It is, however, necessary to add that the experimental results can also be represented by

$$(\chi+K)\,(T+\Delta)=C,$$

where $K=26{\cdot}8\times 10^{-6}$, $\Delta=-29{\cdot}4$, $C=0{\cdot}5455\pm 0{\cdot}0015$,

with an accuracy rather greater with that given by the previous equation. This equation has the further advantage that the value of K corresponds closely to a normal diamagnetism. It is of the same type as the first which we ourselves gave, but the values of K, Δ and C are significantly different.

For Rh^{+++}, which our first study showed to have an almost constant paramagnetism ($\chi_T/\chi_{293}=1$), Cabrera and Fahlenbrach obtain either the equation

$$(\chi-39{\cdot}04\times 10^{-6})\,T=(2742\pm 62)\times 10^{-6}$$

or

$$(\chi-42{\cdot}68\times 10^{-6})\,(T-133{\cdot}1)=(867\pm 47{\cdot}5)\times 10^{-6}.$$

The constants K are of the same order of magnitude, and represent a constant paramagnetism (or show an excess of paramagnetism over diamagnetism). The small value of $(\chi+K)$ accounts for the almost constant value obtained in previous, less precise, experiments. The mean deviations of the individual measured values from those indicated by the equations have almost the same absolute value in both cases. Naturally, as a result of the smaller value of C in the second equation, the relative uncertainty in its value is greater.

Pd^{++} shows a different magnetic behaviour. Below 0° C. it follows approximately the law of Curie with

$$\chi T=(715\pm 17)\times 10^{-6},$$

but the deviations are considerable. Above 0° C., the results can be represented by introducing a K term, giving the equation

$$(\chi-0{\cdot}56\times 10^{-6})\ T=(1789\pm 28)\times 10^{-6}.$$

The results undoubtedly differ appreciably from those obtained in our first experiments.

§ 2. EXPERIMENTAL DETAILS

(i) *Balance and electro-dynamometer.* For the experiments with which this paper deals, we again make use of the non-uniform field method of Faraday, but we have changed the installation. The torsion balance is of the same type as that employed by Cabrera and Fahlenbrach. The displacement is compensated by means of an electrodynamometer of the Helmholtz type. The equilibrium is judged by a lamp-and-scale method, with a graduated scale at a distance of 3 m., by means of which the position of the image of a luminous filament can be estimated to $\frac{1}{10}$ mm. The central coil C_m, figure 1, of the Helmholtz system is fixed to the vertical arm of the movable rod of the balance which carries the specimen, while the fixed coils at right angles to the former are placed above the base of the complete apparatus. The mirror is fixed to the movable rod above C_m, and the latter is prolonged into a glass tube of suitable shape for the avoidance of obstacles between the balance and the magnet. The tube terminates in a mouthpiece with a ground-glass stopper to support the container for the specimen. The centre of gravity of the latter is carefully set above the prolongation of the rod, which is kept horizontal by means of a system of counter-weights at its end. The electric currents are applied independently from two circuits, each of which includes a small number of accumulators, some rheostats for regulating the current, and an invariable resistance ρ in a bath of mineral oil. The value of i is obtained by measuring the difference in potentials between the ends of ρ with an Otto Wolf potentiometer.

For these measurements, as well as for those made with the thermocouples which will be described later, a galvanometer of the Hartman Braun type is used. A commutator enables this to be used for either circuit. The same graduated scale is used as for the determination of the equilibrium of the balance.

(ii) *Electromagnet.* For the production of the magnetic field a special electromagnet is used, the magnetic circuit for which is represented in figure 2. The arm A

is the core of a coil of copper wire insulated with a double covering of cotton and submerged in a bath of mineral oil cooled by a current of water. The two pole pieces are situated one on the arm *A* and the other as an expansion of the parallelogram in front of the former. The magnetic circuit, with the exception of the pole pieces, is constituted by small iron plates firmly fixed in position by means of male and female screws. The plates are 2 mm. in thickness. The yoke has a rectangular section

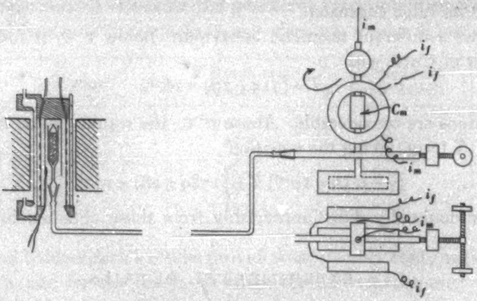

Figure 1. Diagram of apparatus.

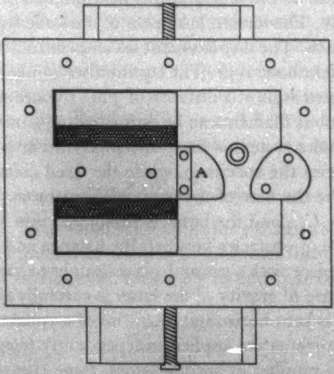

Figure 2. Diagram of electromagnet

measuring 6 × 9 cm. The pole pieces alone are made of homogeneous iron, and their bases have the same area of 6 × 9 cm., while the frontal faces are cylindrical, with a length of 9 cm., the form of the section being such that the dH^2/dx function has a single, sufficiently flat, maximum.

The electromagnet is supported on two rails, along which it can be very slowly driven by an electric motor, in such a way as to bring the specimen just into the region of maximum force.

The current for the magnet, of the order of 13 amperes, is generated by a battery of accumulators, and regulated by continuous variation rheostats. The current can be measured to 0·01 ampere by means of a Siemens Halske precision ammeter. The character of the field-current curve in the range utilized is such that the magnetic field produced can be taken as constant to within less than 0·05 per cent.

(iii) *Oven.* To raise the temperature of the specimen, an oven of the Foëx-Forrer(3) type, connected to the same support as the torsion balance, is used. Its position can be regulated in such a way as to obtain freedom of movement for the mobile system over a range of 7 mm., which corresponds to a deflection of 14 cm. on the scale. To the silver cylinder of the oven a copper-constantan couple is soldered. To estimate the temperature of the specimen, we made a preliminary study of the distribution of temperature along the vertical axis and of the difference between the temperature of the silver cylinder and that of a piece of solder placed at the point corresponding to the centre of gravity of the substance. The couple employed for this purpose is of manganin-constantan, which gives an electromotive force only very slightly inferior to that of the constantan couple, and has the advantage of the low thermal conductivity of the alloys. With this couple, the temperature at different points on the axis of the cylinder is taken and compared with the temperature of the couple soldered to the cylinder at the moment of measurement of the temperature on the axis. The mean temperature along the length of the column occupied by the body can then be determined graphically, and an estimate can be made of the correction which must be applied to the temperature of the silver cylinder to obtain the temperature of the specimen. The measurements of e.m.f. are made with a Cambridge potentiometer.

(iv) *Experimental method.* For each series of measurements the balance and the oven are first adjusted in such a way that the rod is kept horizontal and that its movement remains entirely free. The currents for the magnetic field and the electrodynamometer must be applied at the same time in order to prevent collision of the mobile system with the walls of the oven. The position of the electromagnet corresponding to the maximum value of the product $i_m i_f$ is also found, where i_m and i_f are the currents through the movable and fixed coils (see figure 1). This position remains invariable during the whole series of experiments, although there is sometimes a slight shift of zero of the balance. The correction then necessary is very small and can be determined easily by bringing the system back to the original zero by the action of the electrodynamometer. This procedure gives also the sensitivity of the balance. The value of $i_m i_f$ utilized is the mean of the values obtained by altering the directions of i_m and i_f since it is necessary to eliminate the actions on C_m of the magnetic fields of the electromagnet and of the earth.

Each session of measurements made on the same substance begins with an experiment at room-temperature, always near to 20° C., with the object of keeping a check on any change in the system from one day to another. After this, measurements are made at several temperatures above that of the room, with the temperatures first increasing and then decreasing.

The equation of equilibrium of the balance at a given temperature is

$$K\,(i_m i_f) = \tfrac{1}{2}\chi_s\,[dH^2/dx]_s\,m_s + \tfrac{1}{2}\chi_g\,[dH^2/dx]_g\,m_g + A$$

in which K is the constant of the electrodynamometer. The first term on the right gives the action of the field H upon the body of specific susceptibility χ_s and mass m_s, $[dH^2/dx]_s$ representing an appropriate average of the values of dH^2/dx upon the body. The second term has the same significance in relation to the glass tube in which the body is contained, and the third represents the action upon the rest of the mobile system.

To arrive at a determination of χ_s it is necessary to ascertain the two last terms separately or together. The latter procedure is preferred, and is carried out by making a further series of experiments over the same range of temperatures with an empty tube of the same glass, length, and type as that containing the specimen. Allowance may be made for the very slight residual differences with ample precision. The representative curve of $K(i_m' i_f')$ as a function of T for this series enables one to obtain for each value of T

$$K(i_m i_f) - K(i_m' i_f') = \tfrac{1}{2}\chi_{s,T}\,[dH^2/dx]_s\, m_s,$$

so that

$$\chi_{s,T} = \frac{2K}{m_s\,[dH^2/dx]_s}\,(i_m i_f - i_m' i_f'),$$

giving $\chi_{s,T}$ if the constant factor outside of the parenthesis is known. If it is desired simply to obtain the law of variation with T the simplest method is to take as a standard of reference a temperature of 20° C. or 293° K. very close to that of the room, as we have already pointed out.

We then obtain

$$\frac{\chi_{s,T}}{\chi_{s,293}} = \frac{(i_m i_f - i_m' i_f')_T}{(i_m i_f - i_m' i_f')_{293}}.$$

To obtain now the relative value for the cation $\chi_{c,T}/\chi_{c,293}$ the law of additivity can be used,

$$M\chi_s = c\chi_c + a\chi_a,$$

where M is the molecular weight of the salt, χ_c the susceptibility of the cation, χ_a that corresponding to the anion, and c, a are the numbers of cations and anions which constitute the molecule. Thus we arrive at

$$\frac{\chi_{c,T}}{\chi_{c,293}} = \frac{\chi_{s,T}}{\chi_{s,293}} + \frac{a}{c}\,\frac{\chi_a}{\chi_{c,293}}\left(\frac{\chi_{s,T}}{\chi_{s,293}} - 1\right),$$

where $\chi_{s,T}/\chi_{s,293}$ and $a\chi_a/c\chi_{c,293}$ are known, supposing that $\chi_{s,293}$ has been measured by comparison with the standard substance. We have already stated that for this purpose we adopt the figures of Cabrera and Duperier. For χ_a, the susceptibility of Cl in our case, the Pascal's value of $-20\cdot1 \times 10^{-6}$ has been adopted.

§ 3. EXPERIMENTAL RESULTS AND DISCUSSION

We are here mainly concerned with the experiments made on the salts $OsCl_3$, $IrCl_3$, $PtCl_3$, but in order to establish the connexion between these and the previous experiments of Cabrera and Fahlenbrach, we have started by repeating the study of $RhCl_3$.

We include the $\{1/\chi, T\}$ representation, following a practice customary in former papers, but for the recognition of the law of thermal variation of χ we consider it more suitable to show $\chi_e T$ as a function of T. With the second method of representation of the results, a horizontal straight line indicates that the law of Curie holds, and if the straight line shows any inclination at all this denotes an additional susceptibility independent of T; if the line has a curvature which disappears on addition to T of the constant Δ, then it is the law of Curie-Weiss that is indicated. This is the same representation as that utilized by Cabrera and Fahlenbrach.

To illustrate the general character of the experimental results, and also the proposed method of testing the equations, full details are given for $RhCl_3$ in table 2. The observational data are given in the order in which they were obtained, with the object of showing that the laws deduced do not correspond to systematic errors arising from transformations in the substances. The means and the mean deviations of the values listed under C_1 and C_2 are given at the foot of the corresponding columns.

Table 2. Experimental data for $RhCl_3$ and test of proposed equations

$\alpha_T = (i_m i_f - i_m' i_f')_T / (i_m i_f - i_m' i_f')_{293}$. (See p. 850.)
$\beta_T = \chi_{Rh,T} / \chi_{Rh,293}$.
$C_1 = (\beta_T - 0{\cdot}942)(T/293)$.
$C_2 = (\beta_T - 0{\cdot}942)\{(T/293) - 0{\cdot}355\}$.

T (° K.)	α_T	β_T	C_1	C_2
288·0	0·978	1·007	0·064	0·041
287·5	0·984	1·005	0·062	0·040
326·1	1·049	0·984	0·047	0·032
375·7	1·086	0·972	0·039	0·028
288·5	0·995	1·002	0·059	0·038
415·2	1·086	0·972	0·043	0·032
288·6	1·032	0·990	0·047	0·030
376·5	1·070	0·977	0·045	0·033
453·3	1·086	0·972	0·047	0·036
492·3	1·146	0·953	0·018	0·014
287·5	1·016	0·995	0·052	0·033
548·5	1·108	0·965	0·043	0·035
288·7	0·978	1·007	0·064	0·041
595·0	1·076	0·975	0·068	0·056
288·7	0·962	1·012	0·069	0·044
		Mean	0·051 ±0·011	$0{\cdot}035_5$ $\pm 0{\cdot}006_3$

A graphical representation of the results is shown in figures 3 and 4. The Rh line in figure 3 is clearly a curve, but its average slope is practically zero, which is in accord with the conclusion drawn in our less accurate earlier work. In the $\{\chi T, T\}$ representation an approximately straight line is obtained, figure 4, the slope of which corresponds to $K/\chi_{293°} = -0{\cdot}942$. If the straight-line representation is correct, the fourth column of the table should be constant, except for experimental errors. The distribution of the points on the straight line suggests the possibility that they may be more exactly represented by an equation of the type of $(\chi + K)(T + \Delta) = C$. This

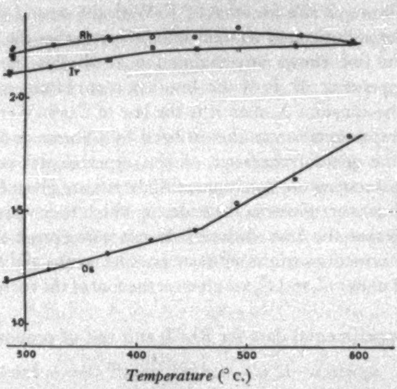

Figure 3. $\{1/\chi, T\}$ graphs for Rh^{+++}, Os^{++} and Ir^{+++}.

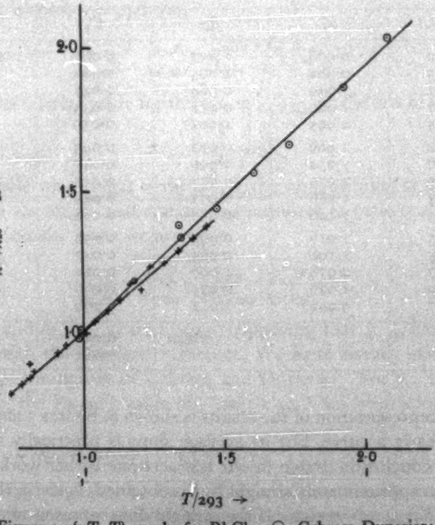

Figure 4. $\{\chi T, T\}$ graphs for $RhCl_3$. ⊙, Cabrera-Duperier; +, Cabrera-Fahlenbrach.

equation is tested in the fifth column, as it appears to represent the results more closely.

Using the equation $(\chi+K)(T+\Delta)=C$ in the form

$$\left(\frac{\chi}{\chi_{293^\circ}}+\frac{K}{\chi_{293^\circ}}\right)\frac{T}{293}+\left(\frac{\chi}{\chi_{293^\circ}}+\frac{K}{\chi_{293^\circ}}\right)\frac{\Delta}{293}=\frac{C}{293\chi_{293^\circ}}$$

we find the following values:

$$K/\chi_{293^\circ}=-0{\cdot}942,\quad \Delta/293=-0{\cdot}355,\quad C/293\chi_{293^\circ}=0{\cdot}035_5\pm0{\cdot}006_2.$$

With the value $\chi_{293^\circ}=+45{\cdot}5\times10^{-6}$ we deduce

$$K=-42{\cdot}9\times10^{-6},\quad \Delta=-104,\quad C=0{\cdot}00047_3,$$

this last number having a probable error of about 17 per cent. The rather large uncertainty in C is connected with its small value, and is not attributable to a wide spreading of the experimental points, as may be seen from the graphs. The corresponding uncertainty in the ratio χ_T/χ_{293} for any temperature in the range investigated is less than 1 per cent.

In the calculation of Δ we have taken the value of K/χ_{293° obtained from the first equation. The residual errors of this latter show that the correction introduced by including Δ is of the second order, and consequently that the value of K will be little effected.

In figure 4 the crosses (+) show the results of Cabrera and Fahlenbrach and the circles (⊙) those obtained in the present experiments. The slight difference of about 2 per cent in the slope is probably within the limits of experimental error. Applying an equation of the same type to the observations of Cabrera and Fahlenbrach, we obtain the values

$$K=-42{\cdot}68\times10^{-6},\quad \Delta=-133{\cdot}1,\quad C=870\times10^{-6}.$$

The two sets of values for K and Δ are in good agreement, while the notable difference for C is tolerable in view of its small value. It can be concluded that the present results are consistent with those obtained by Cabrera and Fahlenbrach for the palladium family.

For $OsCl_2$ measurements were made over a range from 283° to 591° K.

The $\{1/\chi, T\}$ and the $\{\chi T, T\}$ graphs for the cation are shown in figures 3 and 5. The graph in figure 3 suggests the existence of two straight segments, but no significance should be attached to the sharp bend in the graphical representation, as the number of observations is too small to exclude a curve. Nevertheless, in figure 5 it can be seen very clearly that there is a marked difference in the behaviour of Os^{++} below and above 440° K. Below 440° K. the thermal variation is represented by

$$(\chi_{Os}+K)\,T=C_1,$$

with $K=-49{\cdot}8\times10^{-6}$, $C_1=0{\cdot}00964$; and above 440° K. by

$$\chi_{Os}\times T=C_2=0{\cdot}0323.$$

By a treatment similar to that indicated in table 2 for $RhCl_3$, it is found that the mean deviation of the experimental values from those given by the equations corresponds to an uncertainty of 1·8 per cent in C_1 and of 1·1 per cent in C_2. The results for $IrCl_3$, investigated over the temperature range 285° to 591° K., are shown in figures 3 and 5.

For Ir^{+++}, figure 3 suggests either a small curvature or two linear segments, but the $\{\chi T, T\}$ diagram shows that the curvature may be accounted for by including a constant K. The equation representing the thermal variation is

$$(\chi_{Ir}+K)\,T=C,$$

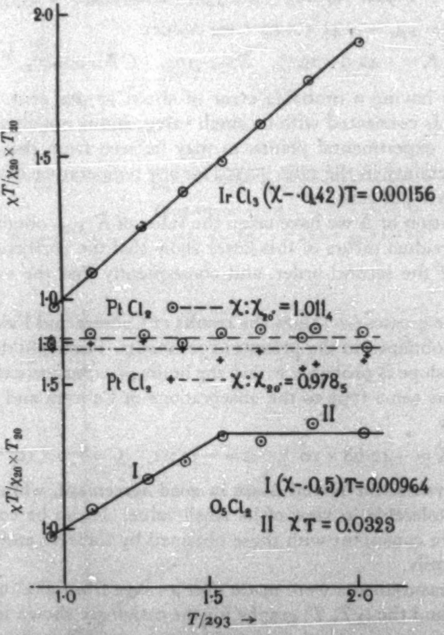

Figure 5. $\{\chi T, T\}$ curves for Os^{++}, Ir^{+++} and Pt^{++}.

in which
$$K=-0{\cdot}889\times 47{\cdot}2\times 10^{-6}=-41{\cdot}96\times 10^{-6},$$
$$C=0{\cdot}1127\times 293\times 47{\cdot}2\times 10^{-6}=0{\cdot}00156,$$

the uncertainty in the C value, estimated as before, being 2·2 per cent.

The curvature of the line sketched in figure 5 is not sufficiently outside the limits of experimental error to justify inclusion of a definite Δ term.

For Pt^{++}, the measurements were difficult owing to the very low value of χ_{Pt}. Measurements were made over the temperature range 283–593° K. The reduced results are shown in figure 5, the circles and crosses corresponding to two different series of measurements made with the empty tube, one preceding and the other following the series with the chloride. Each series suggests a slightly sloping straight line, but the slopes have opposite signs and are almost equal in absolute magnitude, a circumstance which induces us to consider χ_{Pt} constant and to attribute the thermal variations indicated to experimental error for the empty tube. A careful study of the errors suggests a slight diminution of χ_T/χ_{293} with temperature, but this is completely within the experimental error. The whole set of measurements gives a mean value of χ_T/χ_{293} of $0{\cdot}994_7$ with a mean deviation of $1{\cdot}2_8$ per cent.

Naturally, with a χ_{Pt} which is constant we have adopted the value $-17{\cdot}4\times10^{-6}$, obtained by Cabrera and Duperier.

§4. INTERPRETATION

The present results confirm the complexity of the magnetic behaviour of platinum and palladium families, which we pointed out some years ago and which was clearly shown for the palladium family by Cabrera and Fahlenbrach. The most obvious characteristic is the smallness of the susceptibility of these substances as compared with the theoretical values deduced from the electronic configurations of the ions, according to the Hund rules.

Table 3 contains the theoretical values deduced from the ordinarily accepted configurations for these cations(4). The values of the atomic magnetic movement can be immediately obtained from the formulae

$$\mu_{\mathrm{eff}}=g\sqrt{\{J(J+1)\}}\,\mu_B,\quad \mu_{\mathrm{eff}}=2\sqrt{\{S(S+1)\}}\,\mu_B,$$

the first holding when both the orbital and spin moments are effective and the second when the spin moments only are effective. From these values of μ_{eff} those of the atomic susceptibility of the cation are deduced by means of the equation

$$\chi=\frac{\mu^2_{\mathrm{eff}}}{3R(T+\Delta)},$$

in which Δ results from the interactions between the neighbouring atoms in the solid bodies. The results of the calculations refer to the average room-temperature T of 293° K. For the Bohr magneton per gram atom, $Nhe/4\pi mc$, we have adopted the value 5575, which is derived from Millikan's constants for 1938(5).

Table 3

Cation	Configuration	$\mu_{\mathrm{eff}}=2\sqrt{\{S(S+1)\}}\,\mu_B$		$\mu_{\mathrm{eff}}=g\sqrt{\{J(J+1)\}}\,\mu_B$		
		μ_{eff}	χ_c units × 10^{-6}	g	μ_{eff}	χ_c units × 10^{-6}
Ru^{+++}	$^6S_{5/2}$	33000	16570	2	33200	16750
Rh^{+++}	$^5D_4$	27320	15790	$\frac{3}{2}$	37400	29700
Pd^{++}	$^3F_4$	15780	3410	$\frac{5}{4}$	31200	14400
Os^{++}	$^5D_4$	27320	10160	$\frac{3}{2}$	37400	20800
Ir^{+++}	$^5D_4$	27320	10160	$\frac{3}{2}$	37400	20800
Pt^{++}	$^3F_4$	15780	3410	$\frac{5}{4}$	31200	14400

These values of χ_e are not directly comparable with the experimental values, because they refer only to the permanent atomic moments, whereas the experimental values include the constants K and correspond to (χ_e+K).

To make the following discussion clear we give in table 4 the constants obtained in our experiments together with the values deduced from the atomic moments in absolute units and Bohr magnetons. The value of μ_{theor} considered is that corresponding to spin only.

Table 4

Cations	K units $\times 10^{-6}$	Δ	C	μ_{eff} (abs.)	μ_{eff} (B)	$\mu_{\text{eff}}/\mu_{\text{theor}}$
Ru^{+++}	+26·8	− 29·5	0·5455	11640	2·09	0·35
Rh^{+++}	−42·9	−104·0	$0{\cdot}0473\times10^{-2}$	341	$0{\cdot}06_2$	0·013
Pd^{++}	0·0		$0{\cdot}0717\times10^{-2}$	423	$0{\cdot}07_6$	0·027
	+ 0·56		$0{\cdot}01797\times10^{-1}$	751	$0{\cdot}13_5$	0·048
Os^{++}	−49·8		$0{\cdot}0964\times10^{-1}$	1550	$0{\cdot}27_8$	0·057
	0·0		0·03229	2838	$0{\cdot}50_9$	0·104
Ir^{+++}	−42·0		$0{\cdot}0156\times10^{-1}$	624	0·112	0·040

We have already said that two effects contribute to the constant K: the diamagnetism of the cation, corresponding to a general property of all the atoms, and a constant paramagnetism arising from effects involving states additional to the ground state. K is the algebraic sum of both terms. For Pd^{++} above the temperature of 280° K., and for Ru^{+++}, this paramagnetism is inferior to the diamagnetism because $K>0$. The difference is much greater in the second case than in the first, and a comparison of $-K$ with the diamagnetic susceptibility of other ions with a number of electrons very little different from that of Ru^{+++} indicates that the constant corresponds closely to the diamagnetism of the cation itself. We may attribute about the same value of this constant to all the ions of the palladium family as the susceptibility of the neighbouring ions Rb^{+}, Sr^{++}, Ag^{+} have respectively the values -23×10^{-6}, $-18{\cdot}3\times10^{-6}$, -25×10^{-6} which are practically all equal. We may take, as an approximate average value for the palladium triad cations, $\chi=-23\times10^{-6}$. For the platinum family, as the values obtained for Cs^{+}, Ba^{++}, Au^{+} are -35×10^{-6}, -28×10^{-6}, -43×10^{-6}, we may adopt -35×10^{-6} for χ.

It is interesting to note that the values of K for Rh^{+++}, Os^{++} below 440° K. and Ir^{+++}, all with similar configuration, are nearly the same. There is, however, a difficulty in the interpretation suggested for this constant. It is reasonable to expect that the part corresponding to the paramagnetism should be approximately the same for the three cations, but the contribution of the diamagnetism must be greater, numerically, for Os^{++} than for Ir^{+++}. Further, K should be greater numerically for Rh^{+++} than for the other two ions, which is not found. It may be noted that the ratio of the experimental to the theoretical moments is also very different for each of these cations.

Van Vleck explains this decrease as being a consequence of the partial quenching even of spin moments, owing to crystalline field or exchange effects or to the actual formation of complex molecules. That these actions produce a deformation of the atoms which is perceptible in some of the chemical properties was recognized a long time ago by Werner and Pfeiffer, who pointed out the gradual disappearance of typical halide properties in the transition from iron triad to the triads of palladium and platinum.

In view of the difficulty of making any proper estimate of the magnitude of the interaction forces concerned, it seems to us useful to represent the decrease of the atomic susceptibility as arising from a decrease in the number of ions that in the solid retain the normal configuration. The majority, by interactions of one or the other type, have their orbital spin moments completely quenched. The proportion of effective cations N_{eff} per gram atom may be deduced by comparing the effective and theoretical values of the atomic susceptibility, the expression

$$\chi_{\text{eff}} = \frac{N_{\text{eff}}\,\mu^2}{3R\,(T+\Delta)}$$

being used where no other factor distinct from N_{eff} depends on this number. It is generally accepted that Δ represents effects depending on a crystalline field⁽⁶⁾. If all the cations have the normal configuration, N_{eff} is the Avogadro constant and the other factors remain the same. Writing

$$\frac{N_{\text{eff}}}{N} = \frac{\chi_{\text{eff}}}{\chi}$$

the following shown in table 5 are obtained:

Table 5

	Ru^{+++}	Rh^{+++}	Pd^{++}		Os^{++}		Ir^{+++}
			$T < 280°$K.	$T > 280°$K.	$T < 440°$K.	$T > 440°$K.	
$\chi_{\text{eff}} = \chi + K$	1939×10^{-6}	$2{\cdot}6 \times 10^{-6}$	$2{\cdot}4 \times 10^{-8}$	$6{\cdot}07 \times 10^{-6}$	$32{\cdot}9 \times 10^{-6}$	118×10^{-6}	$5{\cdot}2 \times 10^{-6}$
$\frac{N_{\text{eff}}}{N} = \frac{\chi_{\text{eff}}}{\chi}$	0·116	165×10^{-6}	$7{\cdot}1 \times 10^{-6}$	18×10^{-4}.	32×10^{-4}	116×10^{-4}	51×10^{-9}

We have here utilized for calculations of χ_{eff} the values of the atomic susceptibility which were obtained in our first work and given above, from which it is necessary to subtract the corresponding K. Except for Ru, for which one-tenth of the cations conserve their normal moment, the ratio varies between about one-seven millionth and 1 per cent. If this representation is to be taken as a physical hypothesis, the small fraction could not be supposed to arise from any crystalline or molecular regularity, though it might arise from fluctuations in distribution of thermal origin.

With such an origin, however, N_{eff} should depend on T, which would complicate the function $\chi(T)$.

It has been frequently indicated that the superficial position of the magnetogenic group in the iron, palladium and platinum families of elements is the principal cause of the divergence of their magnetic constants from the theoretical values. Also it is clear that such a divergence would increase in the order in which the families have just been named, and this is in accordance with observation. A comparison of Ru and Os is not straightforward because of their distinct configurations, but the reduction of the paramagnetism of Os as compared with that of Ru is evident. For Rh and Ir the values of K and χ_c are practically equal numerically, and the values are almost the same for the two ions, so that no great significance can be attached to the actual differences in χ_c. For Pd and Pt, also of identical configuration, the normal paramagnetism disappears completely in the second. Within every family the sensitivity to external disturbances increases with the cation atomic number.

Unfortunately, the number of substances is as yet small. There are results only for a few simple salts of Mo which were studied by Tjabbes and Bose, and a small number of complexes measured by Bose and Bhar and also by Guthrie and Bourland. It is impossible to draw general conclusions from such limited data, but it is evident that there is no contradiction with the point of view which has been described.

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THE DISPERSION OF WIRELESS ECHOES FROM THE IONOSPHERE

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ABSTRACT. The dispersion undergone by a short train of wireless waves when it is reflected from the ionosphere is discussed. It is shown that, for all ordinary conditions of observation this dispersion does not give rise to any difficulties in the interpretation of experimental results, as regards either the time of travel or the amplitude of the wave train.

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§1. INTRODUCTION

Within recent years the Breit and Tuve(1) pulse method for the investigation of the electrical structure of the ionosphere has become of universal application. When a wireless transmitter is arranged to emit a short signal, the radiated wave may be reflected by the ionosphere and return as an echo. The time of flight of the echo when measured may be multiplied by the velocity of light to give what is usually referred to as the equivalent or virtual length P' of the echo trajectory. When the method was introduced, the significance of this quantity was examined by Breit and by Appleton(2). A further examination of the meaning of the phase of the received echo was made by Breit(3).

It is one purpose of this paper to apply the methods of group analysis to the problem in order to show under what ionospheric conditions true dispersion of the pulse signal will occur in an observable form. Although the pulse method was initially introduced in order to measure P', it has since been widely employed in experiments in which the total absorption in the ionospheric path of the waves has been investigated. In such a case the amplitude of the echo signal is measured and treated theoretically as being the same as that of an infinite wave train passing through the absorbing medium. The analysis is extended to an examination of the validity of this assumption.

§2. THEORETICAL DISCUSSION OF DISPERSION

The study of the propagation of a disturbance through a dispersive medium is usually based upon a knowledge of the frequency-variation of the phase velocity of infinite wave-trains in that medium. In the classical examples of dispersion that have been exhaustively investigated, the change in the form of the disturbance as it travels through the medium is of primary importance. The investigation of the ionized regions of the earth's atmosphere by wireless methods is necessarily carried out by studying some aspect of the wireless signal after it has been refracted in the upper atmosphere and returned to the earth's surface. Thus in the latter problem the changing form of the disturbance in the medium is of only secondary importance compared with the final form observed at the receiving point. A discussion of the dispersion based upon the phase velocity of the components of the disturbance at different points along the trajectory would involve many assumptions concerning the nature of the dispersive medium. Attention must be confined to the total phase-retardation of the components over the atmospheric path between the source and the receiving point.

In the Breit and Tuve method of investigating the upper atmosphere an electromagnetic wave of angular frequency p_0 is sent upwards towards the ionized regions, the amplitude of this wave being caused to vary in some known manner. A pulse of energy is thus sent upwards from T and refracted downwards to the receiver at R. The group time t_1 of transit of the pulse via the atmospheric path is compared with the transit time t_2 along the direct path. The difference between these times, together with the velocity of light c, gives what is known as the equivalent path difference,

$$P' = c\,(t_1 - t_2) = ct.$$

If T and R are close together, as in the case of vertical incidence of the waves on the ionized regions, the equivalent path is equal to P'. It has been shown by Appleton that the equivalent path P' is related to the optical path P by the relationship

$$P' = P + p\frac{dP}{dp}.$$

The study of the dispersion of the pulse may be undertaken in terms of the frequency variation of the optical path P. In all experimental investigations it is the equivalent path P' that is measured, and although P may be deduced in some cases by a method due to Appleton, there exist at present no experimental data for P. This does not hinder the discussion, for it will be shown that the form of the pulse at R depends essentially upon the equivalent path P'.

Consider a signal sent out from T of the form

$$y_T = A\,(t)\,e^{jp_0t}.$$

This may be represented by the Fourier integral

$$f(t) = y_T = \frac{1}{\pi}\int_0^\infty e^{jpt}\,dp \int_{-\infty}^{+\infty} f(\lambda)\,e^{-jp\lambda}\,d\lambda = \frac{1}{\pi}\int_0^\infty \phi\,(p)\,e^{jpt}\,dp,$$

where
$$\phi\,(p) = \int_{-\infty}^{+\infty} f(\lambda)\,e^{-jp\lambda}\,d\lambda.$$

In this equation y_T may be the electric or the magnetic intensity of the electromagnetic wave leaving the sender. The amplitude $A\,(t)$ of this wave of angular frequency p_0 varies with time in some known manner, the function $A\,(t)$ being usually of some simple form which rises from zero to a sharp maximum and then decays to zero.

In terms of the Fourier Integral, the signal may be visualized as the resultant of an infinite number of infinite wave-trains, each of which will be refracted by the ionized region and returned to the receiver at R with a phase-retardation which will depend upon the optical path P and the velocity of light. Thus at the receiving station R the signal will be of the form

$$y_R=\frac{1}{\pi}\int_0^\infty \phi(p)\, e^{jp(t-P/c)}\, dp.$$

The optical path P may or may not be a function of frequency. If it is not then

$$y_R=A\,(t-P_0'/c)\, e^{jp_0(t-P_0/c)},$$

since in this case the equivalent path and the optical path are equal for all frequencies. The form of the pulse at R is then exactly the same as that of the pulse sent out from T, but will occur at the retarded time $(t-P'/c)$.

The same result is approximately true for the case in which P' is not rapidly varying with respect to p. A pulse of the form visualized above will have an effective time of duration, Δt, which will of course depend upon the amplitude function $A\,(t)$. A result well known in group analysis is that in such a case the pulse will be adequately represented by the integral y_T where

$$y_T=\frac{1}{\pi}\int_{p_0-\frac{1}{2}\Delta p}^{p_0+\frac{1}{2}\Delta p} \phi(p)\, e^{jpt}\, dp.$$

In this integral the components of angular frequency lying between $(p_0-\frac{1}{2}\Delta p)$ and $(p_0+\frac{1}{2}\Delta p)$ alone have appreciable amplitude. Moreover, it can be shown that

$$\Delta p\, \Delta t \doteqdot 1.$$

Assuming that pP can be expanded in the Taylor series

$$\begin{aligned} pP &= p_0P_0-(p-p_0)\frac{d}{dp}(pP)_0+\frac{(p-p_0)^2}{2}\frac{d^2}{dp^2}(pP)_0+\dots \\ &= p_0P_0-(p-p_0)\,P_0'+\frac{(p-p_0)^2}{2}\frac{dP_0'}{dp}+\dots, \end{aligned}$$

then the above result follows on substituting this series in the integral, after neglecting the second and higher powers in $(p-p_0)$. This is equivalent to the statement that the sum

$$\frac{\Delta p^2}{2}\frac{dP_0'}{dp}+\frac{(\Delta p)^3}{6}\frac{d^2P_0'}{dp^2}+\dots$$

is small compared with the first two terms of the series.

In nearly all practical cases the series is rapidly convergent. The optical and equivalent paths P and P' are often of the same order of magnitude, and since $p_0 \gg \Delta p$ the first term is usually much greater than the second. To test the validity of the assumption that the third term may be neglected, note that this assumption amounts to

$$\frac{dP_0'}{dp} \ll \frac{2P_0'}{\Delta p},$$

or in terms of the frequency

$$\frac{dP_0'}{df} \ll \frac{2P_0'}{\Delta f}.$$

In practice P_0' would, for F region, be of the order 300 to 800 km. Now with the pulse signals commonly employed, Δf will be approximately 10^4 c./sec. or 0·01 Mc./sec. Thus dP_0'/df must be much less than 10^5 km. per Mc./sec., a condition which as a rule is adequately fulfilled in practice.

It may be concluded therefore that the form of the pulse at R will be almost exactly similar to that sent out at T unless P' is changing very rapidly with respect to p. Distortion may be observed when the frequency of the signal is very near the critical frequency for the ionized region from which reflection is taking place, for the equivalent path may then vary very rapidly with p.

The physical picture of dispersion in this case must be as follows. The Lorentz theory of wave-propagation indicates that an infinite wave-train will be reflected at a level in the ionized region at which a given value of the ionic density exists. Since this level is a function of wave-frequency in a region in which the density is increasing with height, it is obvious that dispersion in the present sense will occur when there is a significant difference in the level of reflection of the component frequencies of the pulse signal. This reveals itself, as is shown above, at times when the equivalent path is a rapidly varying function of frequency.

In order to obtain some idea of the effect of rapidly varying equivalent path on the form of the pulse at R, Kelvin's method of stationary phase may be used. The signal at R is

$$y_R=\frac{1}{\pi}\int_0^\infty \phi(p)e^{jp(t-P/c)}\,dp.$$

Since P is a function of the angular frequency p, it is assumed that the total angle of the waves $p\,(t-P/c)$ may be expanded in the Taylor series

$$p\ \ t-\frac{P}{c}\Big)=p_1\left(t-\frac{P_1}{c}\right)+(p-p_1)\left(t-\frac{P_1'}{c}\right)-\tfrac{1}{2}(p-p_1)^2\left\{\frac{1}{c}\,\frac{dP_1'}{dp}\right\}+\dots.$$

The predominant frequency p_1 at any time is defined as that frequency for which the phase of the component waves is stationary with respect to p. The predominant frequency p_1 is therefore determined by

$$\frac{d}{dp}\left(pt-\frac{pP}{c}\right)_1=t-\frac{P_1'}{c}=0,$$

where

$$P_1'=\frac{d}{dp}\,(pP)_1.$$

If we neglect the powers of $(p-p_1)$ higher than the second, the above series when substituted in the Fourier integral gives

$$y_R=\frac{1}{\pi}\int_0^\infty \phi\,(p)\,e^{j\{p_1(t-P_1/c)-\frac{1}{2}(p-p_1)^2c^{-1}\,dP_1'/dp\}}\,dp,$$

and to the first approximation therefore

$$y_R=\frac{1}{2\pi}\left\{\frac{2c}{dP_1'/dp}\right\}^{\frac{1}{2}}\phi\,(p_1)\,e^{j\{p_1(t-P_1/c)\pm\pi/4\}}.$$

If the equivalent path P' is known in terms of p_1, then p_1 may be found as a function of time.

For example, it may be assumed that P' for a limited frequency range varies linearly with p,

$$P' = a + bp.$$

The retarded time at which p_1 is the predominant frequency is then given by

$$p_1 = \frac{c\,[t] - a}{b}.$$

Now for purpose of illustration it may be assumed that the original pulse is of the form

$$y_T = \frac{\alpha^2}{\alpha^2 + t^2} e^{jp_0 t},$$

in which case the $\phi(p)$ will have the form

$$\phi(p) \doteqdot \pi\alpha e^{-\alpha\,|p_0 - p|}.$$

Substitution gives

$$|y_R| = \frac{\alpha}{2}\left\{\frac{2c}{b}\right\}^{\frac{1}{2}} \exp\left\{-\frac{\alpha c}{b}\,|[t_0] - [t]|\right\}.$$

It is evident therefore that when the slope of the P'/p curve is small, the time interval between the predominance of p_0 and neighbouring frequencies p_1 will be very short. If, however, P' is changing rapidly with p, the pulse-length at the receiver may be considerably increased if the receiver is capable of reproducing it.

In practice, pulse-receivers are usually designed to accept a certain band-width in frequency. The effect of the dispersed pulse on such receivers is interesting. It has been shown above that the frequency p_1 is predominant at a given retarded time $[t]$; if the centre of the receiver band is set to p_1 the received signal will be observed to rise to a maximum at $[t]$, but if it is set a little higher this higher frequency will become predominant at a later time. With dispersive conditions of this sort a slow increase in the frequency of reception of the receiver will result in the reception of an echo signal for which the retardation time increases with the setting of the receiver. The existence of such an effect, which occurs when the emitted frequency is very near to the critical value of the F_2 region, was first pointed out to me by Mr R. Naismith of the Radio Research Station, Slough. It has been observed frequently with these conditions.

§ 3. ABSORPTION IN THE WAVE-PATH

In the above analysis the medium has been assumed to be free from any absorption for the waves travelling through it. In practice the disturbance always suffers absorption during propagation, and it is necessary to enquire into the possible effect upon the resulting amplitude of the pulse at the receiver.

The component waves represented by the Fourier integral will suffer attenuation of amount dependent upon the distance travelled and upon the absorption coefficient of the medium. At any point in the trajectory therefore the resultant pulse may be represented by

$$y_R = \frac{1}{\pi} \int_0^\infty \phi(p) e^{-S} e^{jp(t - P/c)}\, dp,$$

where S is the integrated absorption coefficient for a component wave-train and is a function of wave-frequency. Writing the phase retardation of the component waves as $(pP/c-jS)$ and assuming expansion in a Taylor series possible, we have

$$\frac{pP}{c}-jS=\left(\frac{p_0P_0}{c}-jS_0\right)+(p-p_0)\left(\frac{P_0'}{c}-j\left(\frac{dS_0}{dp}\right)\right)+\ldots.$$

If the emitted pulse form is to recur at the receiver at the retarded time $(t-P_0'/c)$, then the factor (dS_0/dp) must be negligibly small. If this may be assumed and the higher powers of $(p-p_0)$ neglected, the form of the signal at R will be

$$y_R=A\left(t-\frac{P_0'}{c}\right)e^{-S_0}\,e^{jp_0(t-P_0/c)}.$$

To this degree of approximation therefore the maximum amplitude of the pulse will be attenuated in the same manner as the amplitude of an infinite wave train.

Here again the conditions generally obtaining in practice justify the above assumptions. Even if P_0' is of the order of 1000 km., $P_0'/c \doteqdot 0{\cdot}33\times10^{-2}$ sec. and (dS_0/dp) would have to be much smaller than this to be negligible. Now, for waves reflected by region F and absorbed in region E,

$$S=\int k\,ds \doteqdot \frac{\text{constant}}{p^2},$$

thus

$$\frac{dS}{dp}=-\frac{2}{p}S,$$

and since S is of the order 1 to 3, (dS_0/dp) is thus generally of the order 10^{-6}.

Near the critical frequency of the regions, however, S does change rapidly with wave-frequency, but the factor (dS_0/dp) can be of importance only very near the critical frequency of region F_2 in the same frequency range as that for which P' is also a very rapidly changing function of p.

The assumption that pulse-amplitudes may be used in absorption-measurements appears amply justified.

§4. ACKNOWLEDGEMENT

I wish to thank Dr H. T. Flint for a helpful discussion I have had with him on this subject.

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THE DIURNAL VARIATION OF ABSORPTION OF WIRELESS WAVES

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ABSTRACT. The total absorption of wireless waves reflected from region *F* has been measured over Christchurch, New Zealand. The diurnal variation in this quantity has been investigated under conditions which exclude absorption at the top of the wave trajectory; the results refer to the absorption which is thought to occur in region *E*. The variation has been compared with that given by Appleton's theory of absorption in a simple Chapman region. The practical results are found to agree with the theory, as far as the diurnal variation is concerned, both in summer and in winter. The ratio of the summer absorption to the winter absorption is, however, smaller than it should be according to the theory. This discrepancy is discussed. It is shown also that when the waves are returned from the intense *E* region, instead of region *F*, the absorption is of the same order of magnitude, a fact which indicates that the absorbing zone is beneath region *E* in conformity with Appleton's theory.

§ 1. INTRODUCTION

A WIRELESS signal, emitted vertically upwards, may be received at a receiving station very near the emitter after having been refracted by the ionized regions of the ionosphere. During the transmission of the wave through the ionosphere, its amplitude will be attenuated by the absorbing action of the ionized gases which constitute the ionosphere.

In this paper experiments are described in which the amount of absorption experienced by wireless waves refracted at vertical incidence over Christchurch has been measured. Since this absorption varies throughout the day, it is of interest to investigate the manner in which this variation occurs as the sun rises to its minimum zenith distance and falls towards sunset in the afternoon. The diurnal variation of absorption has been compared with the theoretical law due to Appleton(1). The absorption of wireless waves has been investigated under different conditions by Farmer and Ratcliffe(3), Eckersley(4), White and Brown(5), Appleton(1) and Best and Ratcliffe(2). The latter have investigated the diurnal variation over Cambridge, while Appleton, and White and Brown have shown that in south-east England the annual variation did not obey the theoretical law in 1935 and 1936. In the present instance it is found that the diurnal variation follows the theoretical law, but that the summer-to-winter ratio is smaller than the theoretical ratio.

§ 2. EXPERIMENTAL OBSERVATIONS

The amplitude of the emitted spherical wave decreases in proportion to the reciprocal of the distance travelled, and, in transmission through an absorbing medium, will be attenuated so that the amplitude

$$E \propto \exp(-\int k\,ds),$$

where k is the coefficient of absorption and the integral is to be taken over the trajectory of the wave. It is the purpose of these experiments to measure $\int k\,ds$ for the waves; this, as was first pointed out by Appleton, can be done by making use of waves which suffer a double reflection between the earth and the ionosphere. In such a case, the amplitude of the second reflection is proportional to $\exp(-2\int k\,ds)$; thus a direct comparison of the amplitudes of the first-order and second-order reflections gives $\int k\,ds$.

The signals were emitted from a Breit and Tuve pulse-emitter in the physical laboratory at Canterbury University College and were received in the same room. The output of the receiver was used to deflect a cathode-ray oscillograph; the amplitude of the signal was visually observed by comparison with a scale on the screen of the oscillograph. With the emitting and receiving equipment in the same room, it was not possible to make use of the direct signal as a reference signal. The procedure adopted was as follows. A local oscillator with a fixed coupling to the receiving aerial was used to deflect the oscillograph spot in place of the echo signal, and a calibration chart was drawn showing the oscillograph deflection as a function of the oscillatory current of the oscillator for each frequency and for each setting of the gain-control. With constant coupling, the electromotive force induced in the aerial is proportional to the oscillatory current; the chart thus permitted oscillograph deflections to be expressed as electromotive forces in the aerial. With a fixed aerial system the latter are proportional to the amplitudes of the electric field of the incident waves.

When opportunity offered, the electromotive forces due to first-order and second-order reflections were compared, and these observations were used to calibrate the apparatus to give absolute values of the absorption when only the first reflections are present. For this to be possible, the emitting and receiving systems had to be maintained constant; the output of the emitter was checked by readings of the aerial current, while frequent recalibration of the receiver served as a check in this case.

The amplitude of echo signals is always subject to a random variation due to a number of causes. In order to find the average amplitude each reading of the absorption was based on fifty readings taken at intervals of 5 sec. The arithmetic mean value was taken to be the average amplitude.

Although some absorption of energy takes place at all points along the trajectory of waves reflected from the F region there are known to be, to a first approximation at least, two distinct types of absorption: (1) that which takes place along that part of the trajectory where the refractive index μ is approximately equal to unity, and (2) that which occurs near the top of the trajectory where $\mu \doteqdot 0$.

It is the first type of absorption that is discussed here and, in order to exclude the second, the working frequency must be chosen to be as far as possible away from the critical frequencies of either the *F* or the *E* region.

As no special aerial system was used to differentiate between the ordinary and extraordinary waves, the frequency was adjusted until these two waves were superimposed on the oscillograph screen. The amplitude measured is therefore the average of the two waves. This procedure has certain disadvantages, but since, according to the theory, both components should exhibit the same diurnal and seasonal variation, the practical results may be compared with it without error.

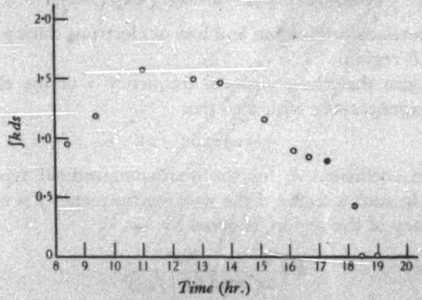

Figure 1. 6 January 1939.

The frequencies used were 5·6, 6·3 and 7·0 Mc./sec. in winter and 6·3, 7·0 and 7·5 Mc./sec. in summer. The winter values were taken between 20 June and 30 July 1938, and the summer values from 5 December 1938 until 6 January 1939. On each day readings were begun between 9 a.m. and 10 a.m., or sometimes earlier in summer, and continued until 5 p.m. approximately. In general, hourly measurements of amplitude were used to calculate $\int k\,ds$. The zenith distance of the sun was calculated in the usual way. In order to illustrate the results for one particular day, the readings taken on 6 January 1939 are shown in figure 1. At 8.30 a.m. N.Z.S.T. the total absorption ($\int k\,ds$) had already acquired the value of 0·95; it rose to a maximum at about 12.30 N.Z.S.T. (noon N.Z.M.T.) and decreased during the afternoon to a small value at about sunset. This curve is typical of the results of each day's measurements. The majority of the results are presented below in a different form.

§ 3. COMPARISON BETWEEN THEORY AND EXPERIMENT

The theory to be tested is based upon the assumption that there is in the path of the waves a region in which the electron-density is produced by solar radiation. Chapman⁽⁹⁾ has shown that, in an atmosphere in which the density decreases exponentially with height, a monochromatic radiation will maintain a rate of production of electrons q such that

$$q = q_0 \exp\{1 - z - \sec\chi \exp(-z)\},$$

where z is a measure of altitude such that

$$z=\frac{h-h_0}{H},$$

where H is the scale-height and h_0 a datum level above the earth. The zenith distance of the sun is χ.

If the rate of production of electrons is very nearly in equilibrium with the rate of disappearance due to recombination, then the electron density N at any time and at altitude z will be given by

$$N=N_0 \exp \tfrac{1}{2} \{1-z-\sec \chi \exp (-z)\}.$$

The equilibrium between production and loss of electrons during the day is known to be true for the E region.

It is assumed also that the collisional frequency ν of the electrons with gas molecules varies exponentially with z so that

$$\nu=\nu_0 \exp (-z).$$

The absorption coefficient k, for the quasi-longitudinal type of propagation which must occur in such a region if the working frequency f is much greater than the critical frequency of the region, is given by

$$k=\frac{\nu}{2c}\cdot\frac{Ne^2}{\pi m\,(f \pm |f_L|)^2}$$

provided that $$\nu^2 \ll 4\pi^2 (f \pm |f_L|)^2.$$

With these assumptions, it has been shown that for such a region

$$\int k\,dz=\frac{4\cdot 13\nu_0 H e^2 N_0}{\pi m c}\,\frac{(\cos \chi)^{\frac{3}{2}}}{(f \pm |f_L|)^2}.$$

The Chapman type of region will therefore cause a total absorption such that

$$\int k\,ds=A\,(\cos \chi)^{\frac{3}{2}},$$

where A is a constant.

The practical measurements of $\int k\,ds$ should therefore show the following features. (1) When $(1+\log \int k\,ds)$ is plotted against $(1+\log \cos \chi)$ the points should lie on a straight line of slope 3/2. Figures 2 and 3 illustrate this method of plotting, which is that used by Best and Ratcliffe. Figure 2 shows the winter values, and figure 3 the summer values. From these curves the slopes have been found and their values are given in table 1.

For the winter results the mean value is 1·80, for summer 1·64, and for all the measurements taken together, 1·72. In view of the difficulty of arriving at a satisfactory average for the amplitude, and the irregularity from day to day, this result is sufficiently close to the theoretical value 1·5 for it to be said that the theory and practice do correspond in this respect.

(2) If $\int k\,ds$ is plotted against $(\cos \chi)^{\frac{3}{2}}$, a straight line through the origin should be obtained if all the absorption in the path of the wave varies with the sun's zenith distance χ in the manner assumed. If, however, there is a region of absorption which

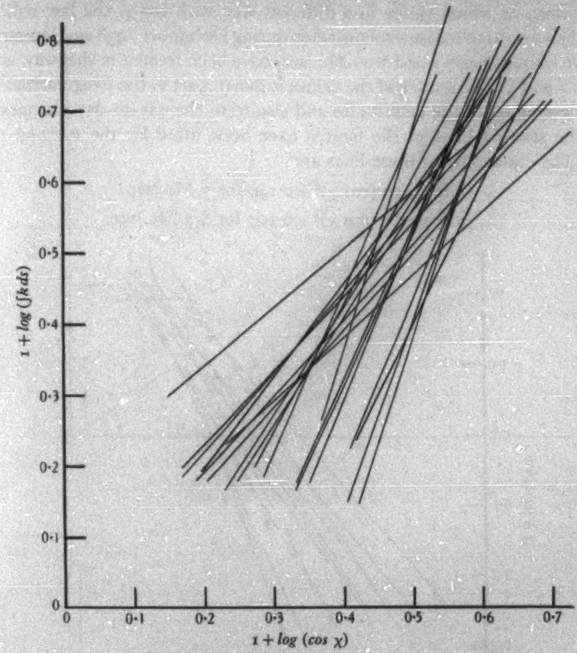

Fig. 2. June and July 1938; winter.

Table 1

Winter		Summer	
Date	Slope	Date	Slope
20 June 1938	2·27	5 December 1938	1·34
22 June	3·04	6 December	1·33
23 June	1·01	9 December	1·60
24 June	2·34	10 December	1·61
2 July	2·70	12 December	2·82
8 July	1·31	14 December	1·38
11 July	0·83	22 December	1·53
12 July	1·07	30 December	1·89
13 July	1·80	4 January 1939	1·16
14 July	1·97	6 January	1·26
17 July	1·53	8 January	2·10
18 July	2·34		
19 July	2·25		
20 July	1·51		
21 July	1·07		

does not vary, or which varies in a different way with cos χ, the law will not be obeyed. The results of measurements taken during December 1938 and January 1939, on the two frequencies 6·3 and 7·00 Mc./sec. have been treated in this way as shown in figures 4 and 5. The scatter of the values is due in part to the irregularities arising from the averaging of the amplitudes and also from the day-to-day changes which occur. The straight lines in the figures have been fitted by the method of least squares. The equations for these lines are

$$\int k\,ds = 1\cdot56\,(\cos\chi)^{\frac{3}{2}} + 0\cdot149 \text{ for 7 Mc./sec.}$$

and

$$\int k\,ds = 1\cdot76\,(\cos\chi)^{\frac{3}{2}} - 0\cdot007 \text{ for 6·3 Mc./sec.}$$

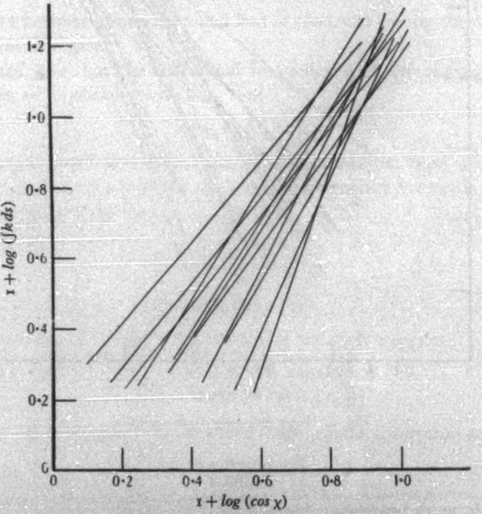

Fig. 3. December 1938 and January 1939; summer.

There are more points at low values of $(\cos\chi)^{\frac{3}{2}}$ in the latter example. In both cases the results lie on a straight line which passes through the origin to the degree of accuracy of the experimental data.

(3) Since there is a large change in the value of cos χ between midsummer and midwinter, a further test of the theory can be made by comparing the summer and winter values of $\int k\,ds$. For this purpose, the mean noon value of this quantity for December–January and for June–July is used. Although the winter values are not symmetrically arranged about 21 June, the values of cos χ vary very slowly at that time of the year, and the error that may arise is not significant. The comparison should yield

$$\frac{\int k\,ds_{\text{(summer)}}}{\int k\,ds_{\text{(winter)}}} = \frac{\sin(\theta+\delta)}{\sin(\theta-\delta)},$$

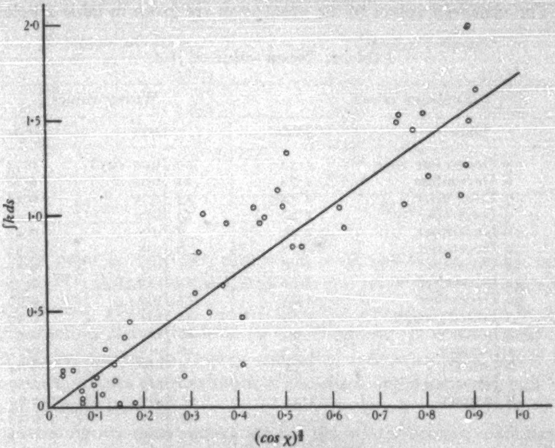

Fig. 4. Frequency, 6·3 Mc./sec.

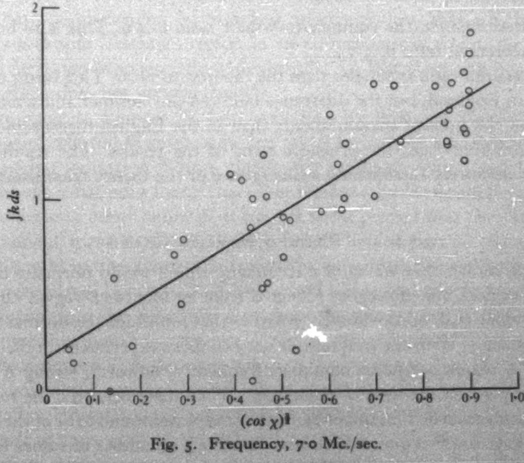

Fig. 5. Frequency, 7·0 Mc./sec.

where θ is the colatitude 46° 30′ of Christchurch and δ is the declination 22° 30′ of the sun. The observed values of the absorption are given in table 2 below.

Table 2. Noon value of $\int k\,ds$

Summer values		Winter values	
Date	$\int k\,ds$	Date	$\int k\,ds$
5 December 1938	1·32	20 June 1938	0·74
6 December	1·78	22 June	0·88
7 December	1·44	23 June	0·44
9 December	2·26	24 June	0·46
10 December	2·05	2 July	0·47
12 December	2·08	8 July	0·54
14 December	1·78	11 July	0·57
22 December	2·00	12 July	0·51
30 December	2·04	13 July	0·69
3 January 1939	1·26	14 July	0·49
4 January	1·90	17 July	0·57
5 January	1·83	18 July	0·72
6 January	1·54	19 July	0·64
7 January	1·13	20 July	0·57
8 January	1·44	21 July	0·69
		22 July	0·55
		23 July	0·53
		25 July	0·70
		30 July	0·38
Mean	1·72	Mean	0·59

From these results, the summer-to-winter ratio is 2·9. This is to be compared with the theoretical ratio of 3·73.

The measured ratio is smaller than the theoretical ratio. This result is similar to that found in England, but the difference between the summer and winter values is much less in the present measurements than in the English measurements, and is not much different from the probable error of the results. The result cannot be regarded as definitely establishing a divergence of the theory from practical observation.

§ 4. INTENSE *E* REFLECTIONS

It is well known that waves of a frequency which would normally be returned from the *F* region, are sometimes reflected from an intense-*E* region characterized by an equivalent path approximately equal to that found for the normal *E* region at lower frequencies. With the frequencies used in these experiments, reflections from the normal *E* region are never obtained; frequently, however, strong *E* reflections were seen in place of the usual *F* reflections. When these occurred the total absorption $\int k\,ds$ was measured. This quantity, in such cases, was found to be of the same order of magnitude as for *F*-region reflections. In table 3 a number of values taken about noon in the winter on radiations of 6·3 Mc./sec. are shown for the two alternatives.

The intense *E* region may allow some partial penetration of the waves, and it is not unreasonable to expect $\int k\,ds$ to be somewhat larger than for *F*-region reflections.

Table 3. Values of $\int k\,ds$

E-region reflection		*F*-region reflection	
0·703	0·682	0·754	0·561
0·966	0·756	0·785	0·719
0·679	0·754	1·090	0·689
0·552		0·861	
0·949		0·531	
		0·560	
		0·452	
Mean 0·755		Mean 0·600	

The important point is, however, that when reflections occur at this very much smaller height ($P' \doteqdot 220$ km. as compared with 500 for *F* region), the total absorption is not materially reduced. This fact indicates that there cannot be important absorption occurring above the level of the *E* region. It is consistent also with Appleton's theory, for the increasing collisional frequency with lower altitudes, which is assumed, causes the maximum of the absorbing zone to lie well below the maximum of electron density(1).

Although the figures given above are for the winter season, a similar result is found for the summer.

§ 5. DISCUSSION OF RESULTS

The above observations of the diurnal variation of absorption show (1) that during the day the absorption is, to a first approximation, that which would be produced by a simple Chapman region in an atmosphere in which the collisional frequency ν decreases exponentially with altitude; (2) that there can be no residual absorption which does not vary as $(\cos\chi)^{\frac{3}{2}}$.

It is reasonable to suppose, as has been done by Best and Ratcliffe, that the Chapman region referred to above is in fact the *E* region. It is known that, during the day, *E*-region ionization follows closely the variation given by the Chapman theory. Although it has been found that other regions (the *D* region, for instance) of ionization sometimes occur below *E*, it has yet to be proved that they have a sufficiently permanent form to be responsible for the major part of the absorption. However, it would not be inconsistent with the experimental evidence if the total absorption were shared between the *E* region and a *D* region of ionization, provided that the absorption in the latter was also proportional to $(\cos\chi)^{\frac{3}{2}}$ during the daytime. It is probable, however, that in a *D* region the collisional frequency ν would not be small compared with the wave angular frequency $(p \pm |p_L|)$, in which case the $(\cos\chi)^{\frac{3}{2}}$ law would not be obeyed. In general, therefore, the evidence of the diurnal variation indicates the *E* region as the probable position of the absorption.

As for lack of agreement of the summer-to-winter ratio with the theory of the annual variation, the following points may be mentioned: (1) Appleton has suggested that it may be due to absorption caused by ionization at a level at which ν^2 is not small compared with $(p \pm |p_L|)^2$. It is true that such absorption would not vary as

$(\cos\chi)^{\frac{3}{2}}$, but it cannot be assumed to occur in explanation of the seasonal variation when the diurnal variation does obey the $(\cos\chi)^{\frac{3}{2}}$ law. (2) It has also been suggested by Appleton as a possibility that group retardation effects due to the F_1 region may have had an appreciable effect in midwinter. The results of White and Brown, which gave approximately the same summer-to-winter ratio for 1935 in England as that obtained by Appleton, would have revealed such an effect. Each result obtained by White and Brown was found from a curve showing $\int k\,ds$ as a function of frequency; in such a curve, the F_1-region group-retardation absorption was immediately obvious and was excluded.

It is considered, therefore, that these suggestions are not likely to account for the observed disagreement. Now the diurnal proportionality of $\int k\,ds$ and $(\cos\chi)^{\frac{3}{2}}$ has a simple meaning in terms of the model atmosphere employed in the theory. The absorption coefficient k, proportional to ν and N, is given by

$$k=\frac{\nu_0}{2c}\,\frac{N_0 e^2}{\pi m\,(f\pm|f_L|)^2}\exp(-z)\exp\tfrac{1}{2}\{1-z-\sec\chi\exp(-z)\}.$$

The maximum value $N_{\max}$ of N occurs when $z=z_0$, where $\exp(-z_0)=\cos\chi$, and has the value $N_0\sqrt{\cos\chi}=N_{\max}$. If the variable is changed so that measurements of distance are made from the level of maximum N, the absorption coefficient takes the form

$$k\propto(\nu_0\cos\chi)\,(N_0\sqrt{\cos\chi})=\nu_{\max}N_{\max}=\nu_0\exp(-z_0).N_0\exp(-z_0/2).$$

The $(\cos\chi)^{\frac{3}{2}}$ law arises from two changes which take place in the model region: (1) the variation as $\sqrt{\cos\chi}$ of the maximum ionization-density, and (2) the change in the level of the maximum ionization in relation to the exponentially distributed values of ν, which gives the term $\cos\chi$. The observed diurnal variation indicates that this model is sufficient for any particular day. The seasonal change can be explained only by postulating a slow variation in the magnitudes (ν_0, H, N_0) involved without a change in the model itself, for the range of $\cos\chi$ on a summer day is greater than the change in this quantity from summer to winter. The change must be slow enough to leave the diurnal results unaffected.

It is known$^{(7)}$ that during 1935–36 the value of N_0 for the E region was increasing. This undoubtedly affected the summer-to-winter ratio of $\int k\,ds$, for it is seen obviously in Appleton's results for the reflection coefficient over this period. The gradual fall of reflection coefficient which was observed is of the right order to be accounted for by the increasing values of N_0. Although this is a slow variation of one of the magnitudes of the model region, it is not of the right type or sufficiently large to account for the anomalous summer-to-winter ratio.

The most satisfactory theory of E-region formation is that due to Wulf and Deming$^{(8)}$. These authors show (1) that the E region may arise from absorption of ultra-violet radiations by molecular oxygen, and (2) that there may be a rapid transition from an atmosphere of oxygen and nitrogen molecules below 80 km. to an atmosphere of nitrogen molecules and oxygen atoms above 100 km. Within this range the molecular oxygen may be distributed very nearly according to an exponential law. The form of the E region will therefore be influenced by the equilibrium of oxygen atoms and oxygen molecules, which in itself is dependent upon the incidence of ultra-violet radiation from the sun. The variation of the molecular oxygen component in this region of the atmosphere with the zenith distance of the sun has not yet been calculated, but it will be of a slow type showing little change over a day.

It may be suggested therefore, that although the model of the absorbing region used above is roughly of the right type to account for the diurnal variation of absorption, no explanation of the seasonal variation can be obtained without some account being taken of the influence of the varying intensity of the solar radiation throughout the year on the distribution of molecular oxygen near the E region.

The theory of the diurnal variation of $\int k\,ds$ indicates that the E region is very nearly of the Chapman form. Experimental evidence of the diurnal and seasonal change in height of this region could be used to check this aspect of the theory in another way. There appear to be no published experimental data of this type.

Note added in proof. Since this paper was written the data for terrestrial magnetism have become available for the days on which experiments were made. It is of importance to mention that all these days were either magnetically quiet or only slightly disturbed.

§ 6. ACKNOWLEDGEMENTS

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ATMOSPHERIC DISTURBANCES DUE TO THUNDERCLOUD DISCHARGES: PART I

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ABSTRACT. The various ways of delineating photographically the random transient changes in the earth's electric field by means of a cathode ray oscillograph are described together with a convenient method of recording the time history of all the disturbances that arise from a thundercloud discharge. An interpretation of these records and observations on the rate of change of field follows. It is found that a positive field change is almost invariably initiated by a volley of pulsations which is succeeded by a sequence of violent changes of similar characteristic structure and indicates the discharge of negative electricity from the cloud to ground by the familiar intermittent lightning flash. The results obtained give the magnitude and nature of the currents involved in the various stages of the flash. Negative field changes observed under the same conditions are recorded as a relatively slow variation of the field carrying minor pulsations and provide evidence relating to the different nature of discharges within the cloud. Following an analysis of the field change due to a component stroke of the intermittent flash an explanation is given to account for the rate of dispersal of charge in the thundercloud towards the end of the lightning stroke which requires a conductivity of the order of 10^2 e.s.u. The records obtained at great distances from the storm centre show that the strongly radiated pulse from each lightning stroke is accompanied by a succession of echoes due to multiple reflections between the ionosphere and the ground.

§ 1. INTRODUCTORY

The recent progress made in the study of the transient changes in the earth's electric field that accompany thundercloud discharges is due to the use of the cathode-ray oscillograph as the recording instrument. An automatic method of recording photographically the wave-form of the atmospheric disturbances as delineated on the oscillograph screen was described some time ago(1) and the records obtained have revealed some remarkable facts relating to the nature of the field change due to a lightning stroke. An account of these observations has already been published(2).

It was found that a single discharge took place in three stages each of which caused a particular type of field change. At the same time it soon became evident that the complete discharge was effected by the rapid succession of a number of similar processes, although from this method of recording it was not possible to make a satisfactory comparison between the wave-form of the separate components. Along with these observations a study of the nature of the lightning flash was being made by Schonland and his collaborators in South Africa using a Boys camera to photograph directly the luminous events that accompany the flash. Walter's⁽³⁾ earlier observations by this method were confirmed and extended, the discharge being found to consist frequently of a number of separate strokes following in rapid succession, while each stroke was itself composite.

A correlation of the results obtained by the two methods established beyond doubt that the field changes we were observing were due to lightning flashes, moreover it was possible to infer from the field-change records both the magnitude and duration of the separate electrical processes involved in the lightning stroke.

The present paper continues the discussion begun in the previous publication on the nature of the field change due to lightning. In the first place, by introducing a continuous method of recording, the time history is presented of all the electrical disturbances that arise from a complete discharge, which may last for several seconds. Secondly, a further study of the characteristic disturbance due to a single component stroke has been made by extending the observations on the rate of change of field resulting in particular from very near discharges. The discussion of the records consists in part of an account of the entirely different field changes due to discharges within the cloud, the nature of which can only be inferred from such records of the electrical disturbances that they produce; and in part of an enquiry into the general characteristics of the discharge to ground and the significance of the differences found to exist between the fine structure of the first component and subsequent strokes of the lightning flash in the light of field-change observations. Finally, some peculiarities of more distant atmospheric disturbances, suggesting the influence of the conducting layers of the upper atmosphere, are mentioned.

§ 2. THE METHOD OF RECORDING FIELD CHANGES

The measurement of the temporal variations of the earth's field by means of the cathode ray oscillograph has been described in the previous paper(2). It is sufficient to mention here that the transient field changes vary in intensity from values of the order of 1000 to 10,000 volts per metre for very near thunderclouds to only a few millivolts per metre for the most distant storms recorded and that for recording these small fluctuations the valve amplifier, forming the connecting link between a condenser placed in series with the exposed conductor and the oscillograph, must at all times give a faithful delineation of these on the fluorescent screen. Ideally this means that the apparatus should give an undistorted response to changes of all frequencies from zero upwards. In practice, however, it is found sufficient and convenient to restrict the lower limit to about 10 cycles per second, the upper limit being determined not by the amplifier but entirely by the maximum velocity of the fluorescent spot that it is possible to record photographically with the low voltage type of oscillograph employed. This corresponds to a frequency of about 100 kc./sec.

The field change is generally asymmetric and takes place by a series of steps with superimposed pulsations due to the component discharges. Relatively slow changes occur during the interval between the major lightning strokes.

Now the total potential-difference that can be amplified without distortion is limited, and if this is arranged to correspond to the total nett change of field due to a nearby discharge, the important detail occurring in this nett change during the separate strokes is too small to be analysed on the screen. This difficulty is overcome to some extent by inserting a high resistance leak across the condenser in the aerial circuit to restore the fluorescent spot to the base line between the strokes while at the same time increasing the amplification to ensure a full scale deflection for each discharge. The high-frequency detail of each component is thus recorded but the introduction of such a time constant may suppress altogether the very slow changes; in any case the record obtained of them is seriously distorted and difficult to interpret in terms of an actual field change.

Under these conditions the spurious effects of this time constant are eliminated if the variations of potential across a resistance placed in series with the exposed conductor are observed instead of the potential across the series capacity as used for field change measurements. Provided this resistance is not too large the oscillograph deflections are now proportional to rates of change of field and by integration of this record actual field changes may be calculated directly. The rate of change of field method has been used in this work for this reason as well as on account of its obvious merits in revealing fine detail.

Cathode-ray-oscillograph photography. The choice of a photographic method of recording depends on whether or not a record of the whole disturbance is required with high time resolution of the separate discharges or a record of only a part of the process which may have a duration of only a few microseconds. In either case the photographic problem lies not so much in the fact that the events are random in nature, occurring at times unknown to the observer and beyond his control, but that the background fluorescence of the oscillograph screen makes it impossible to expose a film to this for more than a very short time, the actual time depending on the electron beam velocity used. The following three methods have been found most practicable.

(1) For recording the detailed structure of the field change due to a single discharge and the wave-form of individual atmospherics a linear time base of from 10^{-1} to 10^{-4} sec. duration is used with a Cossor oscillograph, type E. The screen of this instrument has a long afterglow of approximately 0·25 sec., hence the fluorescent trace of the wave-form persists for this time although the actual duration of the field change may be only a few microseconds. It is this delayed image that is recorded on a stationary film (Kodak Supersensitive Panchromatic) subsequently exposed by the electrical effects of the transient itself⁽¹⁾. The exposure time is adjusted to the optimum value to reduce the fogging, due to the background fluorescence of the screen, to a minimum yet maintaining sufficient intensity of the trace. For a camera lens aperture of $f/1{\cdot}9$ and an oscillograph anode voltage of 1500 volts, this time is about 10^{-2} sec. After each exposure the film is moved on automatically and the camera reset for the next exposure.

(2) For recording the whole disturbance, which may last for several seconds, a continuous time base of this duration is required and on a sufficiently open scale to resolve the rapid fluctuations. A continuous film method alone is not practicable as

the random nature of the events to be recorded makes it necessary to leave the film exposed continuously and the film speed must be at least 10 metres per sec. for adequate resolution. Economy in film has been effected by obtaining fine structure detail, as before, on a rapid horizontal time base, but now using an oscillograph screen of material such that the afterglow is negligible and continuously exposed to a film moving at 20 cm. per sec. in the vertical direction. In this way a trace of saw-tooth form is obtained with the wave-form of the field changes superimposed in time order of arrival(4). The duration of each horizontal time stroke is of the order of 5×10^{-3} sec. and a length of 5 cm. on the screen gives a horizontal time resolution of 10^{-4} sec. per mm. and a separation of the time strokes on the film of 1 mm.

(3) For recording on a still more open time scale the revolving-drum method is the most convenient one for a group of phenomena. This method was the first employed in 1931. The field changes cause vertical deflections of the fluorescent spot with the horizontal plates connected to earth and the record is made on a film or paper (Kodak P 25 or R 25) carried on a drum rotating about a vertical spindle driven by a synchronous motor at 3000 r.p.m. With a drum 1 metre in circumference 1 mm. on the film corresponds to a time interval of 20 microseconds.*

§ 3. THE INTERPRETATION OF FIELD-CHANGE RECORDS

The photographic records give the sign and wave-form of the atmospheric disturbances due to the discharge both as regards its magnitude and duration.

Now balloon soundings made by Simpson and Scrase(5) on the distribution of charge within the thundercloud show that the actual conditions existing are far more complicated than those visualized by Wilson's bipolar cloud and consequently little can be deduced simply from a measure of the sign of the field. We must regard the charge as distributed throughout the region in a number of centres, some positively charged and others carrying negative electricity. Furthermore, since we are not here concerned with the magnitude and sign of the resultant field due to these charged regions, their actual mode of distribution has no significance except, perhaps, for its influence on the subsequent discharge. The records give the field change resulting from the discharge of one or more of these centres by a lightning flash, either to ground or within the cloud to a centre of opposite charge or to the upper atmosphere. The simultaneous discharge of two or more centres will naturally produce a very complex record of the field change incapable of useful analysis so that our discussion is based on the more simple cases.

In the case of a lightning flash to earth involving the lowering of a charge Q, the resulting field change at the place of observation is due to the disappearance of the charge Q from the cloud region and the corresponding induced charge on the surface of the ground. The effect of the latter is most conveniently considered by substituting for it the image of Q, the earth being assumed an equipotential surface before and after the discharge; the lightning stroke then involves a change in electric moment of the thundercloud system of $M=2hQ$, where h is the change in effective height of the charge above the ground. The electric field change E due to the destruction of the dipole moment M at a distance r in the equatorial plane of the dipole is given by the theoretical expression

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* Apparatus recently installed consists of an automatic electrical beam trigger working in conjunction with a hard type of cathode ray oscillograph, continuously exposed to a revolving drum camera. In this way the beam is only brought into focus with the incidence of the disturbance; thereby a continuous background fluorescence and the consequent fogging of the photograph film are eliminated.

$$E=\frac{[M]}{r^3}+\frac{1}{cr^2}\left[\frac{dM}{dt}\right]+\frac{1}{c^2r}\left[\frac{d^2M}{dt^2}\right], \qquad \text{......(1)}$$

where the values of the quantities in brackets are the retarded values obtained at a time $(t-r/c)$ and provided the following conditions are satisfied. In the first place, the distance r must be large compared with the length h of the discharge channel. Secondly, it is implied that the magnitude and phase of the current along the channel are constant, in other words the quasi-wave-length of the radiated pulse must be much greater than h. The charge centre and the lightning channel to earth is then analogous to a radiating aerial consisting of a vertical wire with a large capacity at the top and earthed at the lower end. It has been shown, in a comprehensive study of the spreading of electromagnetic waves from such a system(6), that the earthed aerial can be considered as the upper half of a dipole, its image in the earth acting as the lower half. Now our measurements have been made at distances greater than 10 km. from the discharge and for h of the order of 1 km. the first condition is satisfied. Current pulsations having a quasi-period of 50 μsec. are observed, but for these the quasi-wave-length is 15 km. In applying then equation (1) the observed field change E due to the destruction of a cloud charge Q is the resultant of three components E_1, E_2, E_3, where $E=E_1+E_2+E_3$, and for the electric field near the dipole such that the predominating component is E_1 (the electrostatic term) a nett change of field proportional to the charge Q and decreasing as the cube of the distance is produced. The direction of the discharge may be inferred from the sign of this field change, for by regarding the lightning flash as involving the transport of negative charge it follows that a flash to earth results in an increase of a previously existing positive gradient or a diminution or reversal of a previously existing negative gradient, i.e. in a positive change in the value of E and a flash upward a corresponding decrease. We are thus able to differentiate between discharges in these two directions by observing the sign of the nett change of field.

The presence of the component E_3 or radiation term depends on the maximum rates of change of current occurring in the lightning stroke since

$$E_3=\frac{2h}{c^2r}\frac{dI}{dt}$$

and varying inversely as the distance will be the predominating term at great distances. A characteristic of the radiation field change is that its initial and final value is zero. It will readily be appreciated that for conditions such that one part of the discharge produces a marked electrostatic nett change of field and little radiation field, the rate of change of current in other parts may be sufficiently great to produce intense radiation effects and little corresponding nett change for the same distance. This point will be discussed in more detail later; it is sufficient to mention now that records of this kind are frequent and illustrate the somewhat obvious fact that the current variations throughout the various stages of the discharge vary considerably. For intermediate points of observation all three terms must be taken into account, but such records are only of interest in showing the evolution of the atmospheric wave-form(2), and the present measurements are based entirely on the electrostatic and radiation field changes where these predominate for a particular part of the discharge.

The observations fall into two groups, those made at a distance of the order of 10 km. from the discharge, when the electrostatic field predominates, and those made at distances too great for this to be recognized, the observed field change being due to radiation.

In the former case the nett change E_1 is proportional to the change in the electric moment of the charged region tapped by the discharge, or in the particular case of a discharge to ground, to a change in charge Q of that region. The rate of change of field dE/dt for this is therefore proportional to the rate of change of Q or the current I flowing in the discharge channel, provided that the amplitude at any instant is constant over the whole length of the channel, and, so long as this observation is confined to the electrostatic field change, any fluctuations in the current are readily shown by a temporal variation in the value of dE/dt. For the reasons already given in the previous section measurements of rates of change of field have been made near the discharge. If, however, as at great distances, the radiation field predominates, then the field change E is actually proportional to dI/dt and any further differentiation by the dE/dt method merely complicates the record. Similarly those parts of the discharge having violent current fluctuations will cause large radiation changes near and are therefore best measured by the E method.

Subject to these conditions field-change records have been obtained by both the E and dE/dt methods.

§ 4. THE GENERAL NATURE OF THE FIELD CHANGE DUE TO THUNDERCLOUD DISCHARGES

The heterogeneous nature of thundercloud discharges together with the fact that no two lightning flashes are exactly the same greatly adds to the difficulty of analysis of the records. Yet, from a comparison of all the data obtained for a number of years and by disregarding those features which are incidental, certain very definite types of field change are apparent. For example, those disturbances due to discharges between the cloud and ground which produce an increase or positive change in the potential-gradient, are found to differ markedly from the cloud discharges which result in a decrease or negative field change for the same conditions. Moreover, the positive field change almost invariably takes place by a series of brief pulses which when observed sufficiently near the discharge gives an electrostatic nett change of a stepped-like structure. Occasionally negative changes of this type are observed. The grouping of these component discharges and their bearing on distant atmospheric disturbances and the intermittent lightning flash have already been discussed(2); although the earlier method of recording, which was particularly suitable for the delineation of the field change due to the first component of the series, did not permit a satisfactory comparison to be made between this and the succeeding discharges. These difficulties have been overcome by the recording technique used in the present observations, since we have not been limited as hitherto to recording the changes that occur within the time of duration of a rapid time stroke, but have been able to follow, by the continuous method of recording, the complete time history of all the discharges(4).

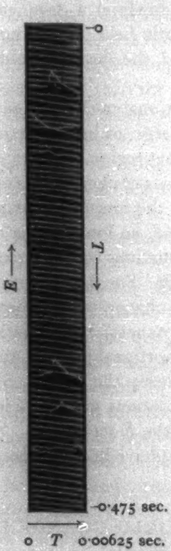

Figure 1.

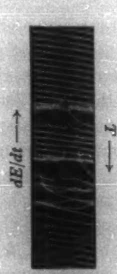

Figure 2 (*a*).

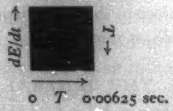

Figure 2 (*b*).

An example of a complete record for this type of atmospheric disturbance taken sufficiently near the discharge for the electrostatic field to be recorded as well as radiation effects, is given in figure 1. It should be noted that the departure of the trace from the direction of the base line denotes a change in the electric moment of the cloud due to a rapid redistribution of charge, while the characteristic decay after each component is due to the apparatus as mentioned in § 2. Positive field-change records of this kind correspond in every way to those to be expected from a lightning flash between the cloud and ground. The successive field changes result from the intermittent nature of the lightning flash as first revealed by Walter(3) (1903) who used a moving camera to photograph the luminous effects of the flash. Similar observations made by Schonland and his collaborators(7) (1935) during the last few years using the Boys camera, besides confirming the intermittent nature of the lightning flash, have provided valuable information as to the nature of the separate component strokes. Since such photographs, which depend on luminosity, are necessarily limited to the more violent parts of the discharge and provide no information of the slower processes occurring between strokes, it was decided to investigate this by observing the rate of change of field for very near discharges; see figure 2 *a*. It is found that the interval between strokes is by no means quiescent. Violent, though much slower, fluctuations in the field occur. The most natural interpretation of these results would thus appear to be that the division of a discharge, of the type described above, into separate strokes is somewhat arbitrary; we can no longer represent all discharges as a series of separate current strokes alone, but as a continuous flow of charge into the channel formed by the first stroke with superimposed violent surges of current during which the channel becomes so intensely luminous as to enable it to be photographed by the moving camera.

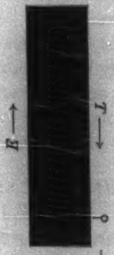

Figure 4 (*a*).

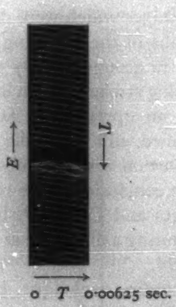

Figure 3.

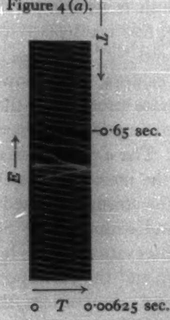

Figure 4 (*b*).

In almost all the observations, where a positive field change of the type shown in figure 1 was recorded, namely one resulting from the intermittent flash between the cloud and ground, the occurrence of a negative field change showed a relatively slow smooth variation of the field often carrying a series of small pulsations which bear a strong resemblance to the pulsations occurring in the predischarge process of the first stroke of a discharge to ground.* Examples of negative field changes are given in figures 3 and 4. That in figure 4 *b* is seen to follow an intermittent discharge to ground, figure 4 *a*, after 0·65 sec.

Negative field changes of this kind are such as would result from an upward flow of negative charge within the cloud, and the significant absence of the rapid succession of sudden and violent nett field changes shows that these cloud discharges are very different from the familiar lightning flash. Schonland, Hodges and Collens(8), from their recent field-change observations, consider this type of discharge to have the same characteristics as the air discharges that have been recorded by the Boys camera(7).

A further variation of the cloud discharge, consisting of a prolonged volley of discharges, has already been reported(4). Here the resulting field change is found to consist of a whole series of radiation pulsations of similar form to those mentioned above but accompanied by a much more gradual electrostatic field change. Such complex volleys of discharges may last for several seconds.

§ 5. THE TYPICAL FIELD CHANGE DUE TO A SINGLE LIGHTNING STROKE

A most striking feature of the field change due to a single component stroke of the lightning flash is its composite nature. It was first shown by Appleton and Chapman(2) that this takes place in three definite stages which were designated as the "*a*", "*b*" and "*c*" portions. Now photographs of the lightning flash, taken with a moving camera, have shown that the flash to ground usually consists of a number of separate strokes, and that a single lightning stroke to earth is made up of a downward-moving leader process followed by an upward and faster-moving return stroke(7). The *a* portion was therefore identified with the leader process, the *b* portion with the main return stroke, and the *c* portion with the subsequent flow of charge into the strongly ionized channel. These conclusions have recently been substantiated and extended by Schonland, Hodges and Collens(8), by means of simultaneous observations of the luminous effects during a discharge as recorded by the Boys camera and the corresponding electric field.

In the present observations the improved recording technique, besides confirming the conclusions made in the previous publication, have revealed further information of these processes and have permitted a comparison to be made between the separate lightning strokes of the complete discharge(4).

The general nature of the field-change for a single lightning stroke, observed at a distance of the order of 10 to 20 km., when the electrostatic field predominates, is illustrated in figure 5. The positive nett change indicates a discharge between the cloud and ground of the type described above. The corresponding record for the rate of change of field is shown in figure 6 (not to scale). Specimen field-change records due to a near discharge are given in figures 1, 7 and 8, while the records of rate of change of field for similar lightning strokes at roughly the same distance are given in figures 2 *a* and 2 *b*. The time-base duration in all the cases chosen is 6·25 msec. reading from left to right.

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* See § 5.

The characteristic atmospheric wave-form record of the field change, observed at great distances from the storm centre, is shown in figure 10. In this case the sensitivity of the apparatus was increased.

(*a*) *The a field change*. It is significant that a relatively slow field change precedes almost every major field change and it was natural to associate this with the leader stroke of the discharge, a conclusion readily substantiated in the case of first leader strokes by a comparison of the duration of the *a* portion of the field change with the value obtained by Schonland, Malan and Collens from actual photographs of the first leader process. This change was sometimes found to be accompanied by a characteristic series of perturbations⁽²⁾ and in the preliminary account of the present work⁽⁴⁾ it was shown that these pulsations seldom accompany the *a* field change of subsequent strokes in the lightning flash. Furthermore they

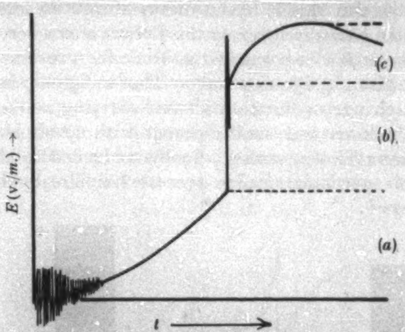

Figure 5. The typical variation of the earth's electric field due to a single lightning stroke, as observed at a short distance from the discharge. The perturbations in part (*a*) of the curve usually only accompany the first stroke of a lightning flash.

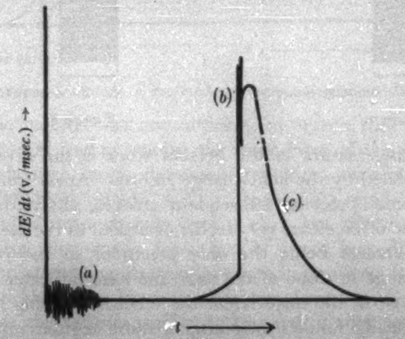

Figure 6. The typical variation in the rate of change of field due to a single lightning stroke where part of the (*a*) portion is too slow to be recorded. Again the initial perturbations are usually absent for strokes after the first. (Cf. figure 5.)

are usually followed immediately by a rapid and intense discharge. They were, therefore, considered to be due to radiation from the successive steps of the stepped-leader predischarge mechanism found by Schonland to initiate a series of lightning strokes. Similar observations and conclusions have been reported more recently by Schonland, Hodges and Collens(8). Although the stepped-leader mechanism is usually followed immediately by the first rapid return stroke of the series, the present records show that this is by no means always the case; sometimes it is unaccompanied by the main discharge, or this follows after a very long time interval. The field change for such a case was that given in the previous communication(4). Other examples are given in figures 1 and 7. That in figure 7 is seen to commence with a fairly rapid change of duration 2 msec. carrying marked radiation effects followed by a much slower and smaller change with no observable perturbations which is, after an interval of 0·012 sec., finally succeeded by the main discharge. The nature of this particular leader process has already been discussed by Schonland and others(8).

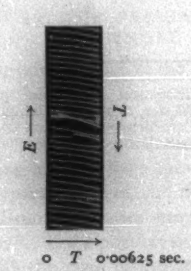

Figure 7.

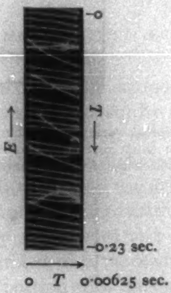

Figure 8.

A most interesting feature of the present work is the diverse nature of the predischarge as revealed by the field-change records. Apart from the fact that the stepped process rarely precedes subsequent strokes, shown by the absence of pulsations, it is quite often either very feeble or absent in the case of first strokes, a long smooth field change being the only precursor to the rapid main stroke, figure 4 *a*. The change in slope of the base line here indicates a gradual positive field change and even in the majority of the records showing the stepped leader this process occasionally follows only after a more regular initial change in the electrical moment of the cloud, figure 8.* It is difficult to visualize anything but a disruptive flow of charge during the initial stages of breakdown with consequent radiation field changes, yet this smooth change suggests a much more regular flow of charge. Under the atmospheric conditions prevailing just before lightning phenomena when the average field strengths in the regions just beneath the cloud are very low(5) it seems evident that considerable space-charge distortion must

* Variations in the more usual stepped-leader process of the predischarge shown by lightning flash photographs have just been reported(9).

initiate breakdown. Such a readjustment of space charge before the main discharge might produce the field change in question which very often shows a redistribution of charge of even greater magnitude than that caused by the main stroke. Whether the effect is due to space-charge movement or to a flow of negative charge from the upper atmosphere to positively charged regions in the upper part of the cloud is not yet clear. It is true that for the more highly conducting regions above the cloud such a current would produce a positive field change of this kind (page 4) and while neutralizing an upper positively charged region would increase the potential-gradient below the cloud to the critical value for initiation of the discharge. A similar process might be taken to account for the long smooth field change often found to precede subsequent strokes, figure 4 *a*, for here again it is difficult to reconcile such a field change with the rapid dart-stroke leader.

The duration of the interval between the pulsations on the record of the stepped-leader process is found to lie between 30 and 100 μsec., but the resolution of the above photographic records was not sufficient to permit this measurement to be made satisfactorily, still less to resolve the wave-form of the separate pulsations. Accordingly records with much increased resolution were obtained by using a high-speed revolving drum camera or alternatively on a rapid time stroke brought into focus and tripped automatically by the electrical effects of the atmospheric disturbance. A projection made from one of these records and showing the structure of the step-streamer pulsations is given in figure 9. It will be seen that a relatively smooth field-change carries pulsations of very similar form and, being common to most of the records studied in this way it suggests some very fundamental process in the first predischarge. As has already been mentioned the relatively smooth departure of the trace from the base line giving a positive field change may be taken to indicate a gradual lowering of negative charge from the cloud centre during the interval between the rapid current pulses, when a sudden redistribution of charge occurs. Each pulse is initiated by a gradual increase in the current followed by the rapid flow which decays finally in the manner shown to a steady value. This wave-form suggests radiation from the rapid current fluctuations separated by intervals of between 30 and 100 μsec. and sometimes more, and for

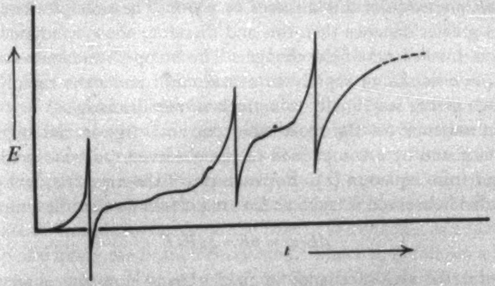

Figure 9. Structure of the field change due to step streamers.

this and the reasons already stated it was first identified by Appleton and Chapman(2, 4) with the stepped streamer.

As regards the rate of change of field resulting from the brief current pulse, the value is found to lie between 4×10^3 v./m.-sec. and 7×10^4 v./m.-sec., the most frequent value being 2×10^4 v./m.-sec. For the rapid return stroke and at distances up to 10 km. it has already been reported that corresponding values of the order of 2×10^7 v./m.-sec. are observed(2, 10), where the change in the cloud moment gives a nett change of field of the order of 700 v./m. in 0·05 msec. For similar conditions the stepped leader resulted in a peak field change of the order of 0·5 v./m. during the rapid current pulse which lasted about 50 μsec. Assuming as a simple approximation that this maximum current pulse is of the form $I = I_0 \sin pt$, starting at time $t = 0$ and ending at time $t = \pi/p$, then π/p is equal to 50 μsec. or the quasi-period T of the radiated pulse is 100 μsec. which corresponds to a quasi-wavelength of 30 km. The radiation and electrostatic field changes will therefore be equal at about one-sixth of this distance or 5 km. The examples cited above were obtained at a greater distance than this and therefore show a radiated pulse superimposed on a smaller nett field change. The stepped streamer current is thus essentially aperiodic, rising rapidly to a maximum and then rapidly decaying in about 50 μsec., giving way finally to a much slower discharge.

The form assumed for the most rapid current surge in the stepped leader is readily substantiated by a comparison of the observed radiation field-change with that calculated from equation (1). For example, if the most frequent peak value of the radiation field observed is taken as E v./m., it follows that the simple expression

$$dE/dt = pE = 2\pi E/T$$

should hold for the rate of change of field. Using now the observed value for $E = 0{\cdot}5$ v./m. and the quasi-period as defined above to be 100 μsec. we find the value of dE/dt equal to 3×10^4 v./m.-sec., which is of exactly the same order of magnitude as that most frequently observed for this part of the discharge.

It is of interest also to determine the peak value of the stepped streamer current. Again applying equation 1 for $E = 0{\cdot}5$ v./m. for the radiation pulse, and remembering that the corresponding maximum rate of change of current

$$dI/dt = I_0 p = I_0 2\pi/T,$$

we find by means of a simple calculation that the peak current value I_0 is of the order of 1000 amp., i.e. between ten and one hundred times greater than the pilot-streamer current and a hundred times less than the return-stroke current. Schonland has estimated that the dart streamer predischarge of subsequent strokes carries a current of from 1000 to 10,000 amp., involving the removal of 1 coulomb of charge in 1·0 to 0·1 msec.(14) which is an essentially similar process to the step streamer. The charge lowered during the main current surge in the step streamer as calculated from the above expression for the current is $I_0 T/\pi$ or 0·03 coulombs, which at least indicates the order of magnitude and shows that if such step streamers alone were effective in causing the large change of moment that precedes the main return stroke, a very large number would be necessary. It is seen, however, that a considerable proportion of the charge is lowered during the long intervals between the rapid current pulses.

(*b*) *The b field change*. At this stage the relatively slow predischarge nett field change gives way to a very abrupt one of almost equal magnitude which is usually completed in about 100 μsec. and most frequently in 50 μsec. and corresponds in every way to the electrical effects to be expected from the intensely luminous main return stroke succeeding the leader process.

It is significant that the nett changes of field due to both processes are of comparable magnitude, and since the time taken by the main return stroke to travel up the channel formed by the leader is about 50 μsec. it is natural to conclude that the charge lowered by the leader and covering the whole channel and its branches is again further lowered to earth by the rapid return stroke. The wave-front of the electric disturbance is extremely sharp (less than 10 μsec.), then falls off and carries complex intense radiation pulsations to which the atmospheric disturbance at a distance is mainly due.

The presence of these purturbations is believed to be due to branching of the channel(2). In this connexion the observations made by Malan and Collens(11) on the luminosity of the channel are of importance. They have shown that the velocity and luminosity decrease as the return stroke progresses in agreement with the observed decrease in the rate of change of field or current flowing in the channel, as the abrupt *b* portion merges into the more gradual *c* portion. Further, Malan and Collens have found that, as branches are reached by the upward-moving streamer, a violent increase of luminosity occurs at the base of the channel as a further streamer progresses up and along the branch concerned. There thus follows a rapid succession of component strokes. The most frequent durations of the first four components are less than 10 μsec. for the first, 20, 50 and 100 μsec., with times of starting, as measured from the time the first component leaves the lower end of the channel, of 0, 25, 70, and 150 to 500 μsec. In view of these observations a more detailed analysis has been attempted of the fine structure found to exist on the *b* portion of the field-change record. Again, as has previously been mentioned, increased photographic resolution of component pulses was obtained by using the revolving-drum method for both E and dE/dt recording. The fine structure was found to be very complex, which is not surprising in view of the heterogeneous nature of the current surges and discontinuous lengthening of the path. Numerous current pulsations of durations from 10 to 250 μsec. occur, with times of separation as short as 10 μsec., but no such simple classification as that described above has been found possible. Quite often the longer current pulses carry minor oscillations of a few μsec. duration. As has been found by Malan and Collens(11) current surges may continue after the return stroke has reached the cloud, but usually they are of very much greater quasi-period ranging from 10^2 to 10^3 μsec. or more, and in spite of the longer channel being a more effective radiator they therefore cause less disturbance at a distance than the initial rapid return stroke with its vigorous components.

An estimate of the order of magnitude of the current in the return stroke may be made on the assumption that the typical value 10^7 v./m.-sec. for the rate of change of field at 10 km. is due to a main-current surge throughout the whole length of the channel. Taking this to be 2 km. the corresponding current would be 300,000 amp. which if flowing for the time 50 μsec. taken by the return streamer to reach the cloud would mean that the charge lowered was 15 coulombs. Now it has been shown by Wilson(12) that the quantity of electricity discharged in an average lightning flash, and this may consist of several separate strokes, is 20 coulombs; hence a single lightning stroke of the above magnitude is sufficient to account for almost complete discharge, the remaining charge being lowered by the final stage.

(*c*) *The c field change.* The completion of the lightning stroke by the flow of charge from the cloud down the strongly ionized channel left after this second process is identified by the *c* portion in the field change, which, in the simple type of discharge now under consideration, indicates a gradual falling off of the current and must correspond to a decay in the luminosity of the channel. The photograph of this shows that the luminosity is intensely bright for a very short time (of the order of 40 μsec.), then decays relatively slowly and is generally absent or either too weak to be recorded before the next component stroke of the lightning flash. It has been found that after the return stroke has reached the cloud the luminosity of the channel sometimes rises and falls(11). Reference has already been made to these complex discharges in the light of the field change they produce. The fact that the next component stroke follows the same track indicates the persistence of ionization. It follows that any measurement of the duration of this stage from the luminosity is uncertain, since it depends on photographic sensitivity. Similarly, for the reasons already mentioned, measurements of the slow passage of charge from field-change records are impracticable. This is clearly shown by the dE/dt records, which are free from the limitations impressed by the time constants of the apparatus, and when made near the discharge show a characteristic decay of the current in a time of the order of 1000 μsec., figures 2 *a* and 2 *b*. This compares with the most frequent duration of the luminosity in the channel(11), but sometimes this is followed by much slower current fluctuations which quite often persist until the next component stroke, figure 2 *a*, and would be responsible for only very feeble luminosity.

The form of the current-decay curve suggests that the current is drawn initially from a wide area. On this view the initial steep portion would correspond in the main to the component of current drawn from centres which are quickly discharged while finally only that from centres controlling the critical field for breakdown are concerned. When we endeavour to analyse the curve in this way the form of the lower portion suggests that the current density results from a time rate of decrease in the charge density proportional to the charge, that is,

$$-d\rho/dt = \alpha\rho,$$

where $d\rho$ is the decrease in the charge-density in time dt and α is a constant characteristic of the medium, the significance of which will be discussed later. The integration of this equation gives

$$\rho = \rho_0 \epsilon^{-\alpha t},$$

where ρ_0 is the charge density at any given instant and ρ the corresponding value after t sec. It is assumed in the integration that α is independent of ρ. The quantity θ, equal to $1/\alpha$, may be termed the "time factor controlling the rate of decay". The significance of this factor and an explanation of this form of the current curve will now be given.

In order to understand something of the nature of the decaying current in the discharge channel after its rapid formation by the leader and return stroke processes, it is first necessary to consider the conditions existing during the discharge. We may assume the discharge to be initiated in the most intense part of the field within the cloud due to the charged regions of the thundercloud. The charges producing the field are distributed through a volume and are carried initially entirely by water drops and large ions. The actual conditions existing during a discharge are open to conjecture, but it seems quite reasonable to expect that the conductivity of the cloud, originally smaller than the air outside owing to the combination of ions with water particles, is now considerably increased by electrical breakdown of the air and such processes of ionization as those discussed by Mackay(13). Under these conditions we shall assume the instantaneous value of the volume density of charge to be ρ. With every point in this region we must associate an electric field intensity E. If the field intensity is less than the critical value at which breakdown occurs, then the discharge will cease. Now for the volume distribution of charge the density of which is ρ the divergence in the flux of electric force is, according to Maxwell's theory, given by

$$K \operatorname{div} E = 4\pi\rho, \qquad \text{......(2)}$$

where K is the effective dielectric constant of the medium.

Equilibrium is not established again until the critical value of the field, and with this a corresponding reduction of the charge density, is reached. While this process of neutralization is going on, current flows through the region and down the discharge channel. If now we associate this current with a time rate of change of the charge density, then, by the equation of continuity, we have

$$\operatorname{div} i = -\frac{\partial \rho}{\partial t},$$

where i is the current density. But

$$i = \sigma E,$$

and defines σ the specific conductivity of the medium; hence

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} (\sigma E). \qquad \text{......(3)}$$

If we regard σ and K as remaining sensibly constant during the brief period now under consideration, the equations (2) and (3) give

$$\frac{\partial\rho}{\partial t} = -\frac{4\pi\sigma}{K}\rho,$$

from which by integration, we have

$$\rho = \rho_0 \epsilon^{-t/\theta},$$

where ρ is the charge density at time t, ρ_0 its corresponding value at $t=0$, and θ, equal to $K/4\pi\sigma$, is the time taken for the charge density to fall to $1/\epsilon$ of its original value.

From this expression it is clear that, for the region tapped by the highly conducting discharge channel, the time rate of dispersal of the charge Q left after the combined action of the leader and return stroke where

$$Q = \int_0^v \rho\, dv$$

takes place according to a logarithmic law. Since at short distances from the discharge the nett field change is in this case proportional to the change in charge of the region, and if, as we have supposed, K and σ remain constant during part of the change, the field for this should decay exponentially during the process.

A field change of this type is indicated by the form of the *c* portion in the record, and for the reasons already mentioned the comparison is best made from observations on the rate of change of field, which is proportional to the rate of change of charge. Hence the curve for this part of the dE/dt record is also of the form

$$dE/dt = k\epsilon^{-t/\theta},$$

where k is a constant.

In order to test this theory the relevant portion of the dE/dt records was plotted by projection on an arbitrary scale. Since the variation of log (dE/dt) with time should be linear these results are replotted accordingly. The curve is very nearly a straight line; hence the initial assumption, as regards the constancy of the dielectric constant and conductivity of the charged region during this stage of the discharge, seems to be valid. Furthermore, the field change due to this distribution is of a type approximating very closely to that characteristic of a dipole similarly discharged, the concept used to interpret the record. It follows, therefore, that the strongly ionized channel left by the predischarge and return-stroke processes carries a current which decays for a time exponentially and whose amplitude at any instant is constant along the whole length of the channel.

The value of θ obtained from the slope is of the order of 10^{-3} sec. To make an estimate of the conductivity required to account for this rate of dispersal of charge in the thundercloud towards the end of the lightning stroke, we have

$$\theta = K/4\pi\sigma.$$

The value of K is most uncertain, but there does not seem to be any reason for supposing that it differs appreciably from unity. Taking a value of unity for K we find that the conductivity σ is of the order of 10^2 e.s.u.

§ 6. THE FIELD CHANGE OBSERVED AT GREATER DISTANCES

From the observations made near the discharge as described above for a single stroke of the lightning flash, it is possible to infer the nature of the field change to be expected at greater distances. The magnitude of the field change may be calculated from the expression for E, equation (1). It is seen from this that the predominating field change at great distances will be due to radiation from the stepped leader and rapid return stroke when the moving charge is subjected to marked acceleration or retardation, while the effect of the subsequent flow of charge may readily be calculated. Remembering that $M=2hQ$, we find that the expression for the moment of the charge lowered during this process is $M=M_0e^{-\alpha t}$ where $\alpha=1/\theta$ so that the equation for E may now be written

$$E=M_0(1/r^3-\alpha/cr^2+\alpha^2/c^2r)\,e^{-\alpha(t-r/c)}.$$

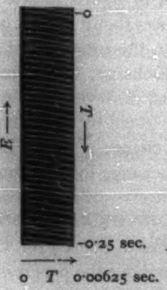

Figure 10.

The distance r_0 at which the radiation field or induction field becomes equal in magnitude to the electrostatic field is c/α. To make an estimate of this we assume the most frequent value of $\theta=10^{-3}$ sec., thus $r_0=300$ km. As most of the atmospheric wave forms have been obtained well within this distance such slow field changes can only be effective in producing an electrostatic field change. This effect, however, falls off inversely as the cube of the distance and is therefore quite often too small to be recorded of great distances, the main atmospheric disturbance being due to radiation from the rapid current fluctuations in the b portion.

On this view the records made at great distances should permit the calculation of the duration and current variations of the lightning stroke. In point of fact, however, this is not possible, for the atmospheric wave-forms due to distant discharges show a great diversity of form, such that it is impossible to correlate them directly with the current fluctuations found to exist from nearby records. While atmospherics do occur in groups showing features exactly similar, as regards number and time interval between the component strokes, to those observed in the case of nett field changes and lightning-flash photographs(2, 8), their wave-forms are found to show a very remarkable series of pulses of decreasing amplitude and lasting for a time very much greater than that occupied by the major current fluctuations in the lightning stroke.

An example of a distant radiation field change is reproduced in figure 10. It shows a group of distant atmospheric disturbances due to an intermittent lightning flash, and serves to illustrate the kind of distortion produced in the atmospheric wave-form during propagation from the seat of the disturbance.*

The nature of the characteristic series of pulses and other effects, observed at still greater distances(2, 15), strongly indicates the influence of the conducting layers of the upper atmosphere on the propagation of atmospherics. The regular sequence of the pulses and their decreasing amplitude are due to successive reflections of the intense radiated pulse from the return stroke between an ionized layer and the ground.† Further support for the view that this effect is due to ionospheric influences is obtained from the fact that echoes of this kind are most readily observed after sunset. The record given in figure 10 was taken about midnight. These points, together with the differences found to exist between the above type of distant atmospheric disturbance, due to the intermittent lightning flash to ground, and those due to other modes of discharge, will form the subject of discussion in a future publication. It is obvious, however, that any view implying causes of this kind could be most conclusively tested by simultaneous observations at two widely separated stations, of the rate of change of field (electrostatic) very near a discharge and the resultant radiation field change for the same discharge at the more distant point of observation.

§ 7. ACKNOWLEDGEMENTS

In conclusion the author wishes to thank Prof. C. D. Ellis and Dr H. T. Flint for their continual interest throughout the progress of this work.

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* A similar example for a single stroke but also showing radiation from the predischarge was given in the previous publication.(4)

† Effects of this kind have been observed by Prof. T. H. Laby and his co-workers in Australia(16). Further, it is interesting to note from figure 10 that the separate echo pulses of the first strokes give way later to an oscillatory structure of the type already discussed(2), indicating a change in the reflection characteristics of the ionosphere during the discharge. The effective height of the reflecting layer and distance of the storm centre are readily calculated. Direction-finding apparatus is now being installed to give the location of a thunderstorm region, from observations at a single station.

REVIEWS OF BOOKS

Introductory College Physics, by OSWALD BLACKWOOD, Ph.D. Pp. x+487. (New York: John Wiley and Sons, Inc.; London: Chapman and Hall, Ltd.) 17*s*. 6*d*.

Prof. Blackwood has collected into book form the substance of a course of lectures on physics delivered to non-technical students. Much of his illustrative material is chosen from sources familiar to all, such as motor-cars, sport, etc., but in addition there are clear and full descriptions of the principles of physics both in their general application and in their special application to subjects such as refrigeration, weather and its prediction, acoustics of buildings, etc. There is a short section on "The new physics". In some future edition it should be possible to record the names of Bragg, Rutherford, Thomson and Wilson. Summaries and questions are appended to each chapter and many numerical examples are worked out fully.

The aim of the author is to instil a knowledge of physics into the minds of students who in after life may be business men, clergymen, lawyers, or the like, and thus to enable them to realize to what a great extent physics guides progress in the social world. So well has he carried out this object that one may hope that it will not only be the budding parson or politician who will benefit, but that, as in cases of which one has heard where parents, having made presents of meccano sets to their offspring, shut themselves away with the pieces, so the older members of the families to which the students belong may find that, with such a readable survey available, it is never too late or too difficult to learn something of the science which has given so much to mankind.

J. H. B.

Advanced Experiments in Practical Physics, by J. E. CALTHROP. Pp. xix+121. (London: William Heinemann.) 8*s*. 6*d*.

This book fills a distinct gap by supplying at a moderate price a description of a series of experiments suitable for an honours degree course in physics. The subject is well covered by the 48 experiments or small groups of experiments included, and though some of them require expensive apparatus—for example that on the Zeeman effect requires an echelon grating—many make use of only the simplest material, and most of them can be carried out in any reasonably well equipped laboratory. The descriptions, while full enough for the serious student, are not tediously detailed and occasionally they wisely assume help from a demonstrator; for instance "the manipulation of a Jamin interferometer is best explained by a demonstrator". Useful references for further reading are given at the end of many of the sections. Altogether the book is one which the B.Sc. physics student would be well advised to buy and use, instead of expecting to be provided with detailed instructions for the exact experiment he does.

A paragraph in the introduction, which recommends the student to equip himself with a pocket screw-driver, safety-razor blade, and soldering outfit, will be read with misgiving by some laboratory stewards. They would prefer to have the right to search for and remove all tools before the student is allowed to enter the laboratory, but, although with some trepidation, the reviewer would support Mr Calthrop.

J. N.

Demonstration Experiments in Physics, edited by R. M. Sutton, Ph.D. Pp. viii + 545. (London: McGraw-Hill Publishing Co., 1938.) 25*s*.

This compendium, by a team of American physicists, is well described in the introduction as a "cook-book". It is a well planned book of carefully classified recipes for the preparation of nearly twelve hundred demonstration experiments, including, in addition to those found in most degree courses, many outside the ordinary beaten track which the active teacher will be glad to meet, and from which he will be able to make selections appropriate to his special needs and resources. Besides sections on mechanics and properties of matter, wave motion and sound, heat, magnetism and electricity, and light, one on atomic and electronic physics usefully increases the range covered. The collection will prove a welcome addition to any demonstrator's book shelf.

B. M. N.

Supersonics; The Science of Inaudible Sounds, by R. W. Wood. Pp. viii + 158. (Brown University, Providence, R.I.) $2·00

This little book contains the three Culver Lectures given at Brown University in 1937 on the first occasion on which the lectures were devoted to a physical subject. The choice of Prof. R. W. Wood to give lectures on supersonics was an admirable one, since his name is associated with pioneer work on the more spectacular effects of these radiations, work which has inspired many researches in physical chemistry and biology and even led to industrial applications. This part of the text consists largely of quotations from the famous paper published by Wood and Loomis in 1927. Interesting sidelights on the history of this new branch of physics are given in respect of Langevin's epoch-making experiments made in 1917; it appears that many of the effects subsequently discovered by other workers, including some by Wood himself, had already been observed in the Toulon experiments, but remained unrecorded because Langevin's chief concern was to perfect a submarine-detecting apparatus, which he succeeded in doing just as the War finished. The booklet can be recommended as an easy and readable introduction to inaudible sound.

E. G. R.

The Principles of Electricity and Magnetism, by G. P. Harnwell. Pp. 619. (London: McGraw-Hill Publishing Co., 1938.) 30*s*.

Physicists who have taken up industrial research have often remarked on the discontinuity experienced on passing from the electricity and magnetism of the classical textbooks to those of the modern electrical laboratory. The point of view and method of attack are so changed that it is sometimes difficult to recognize the subject. There is, therefore, room for a volume which links together a systematic account of fundamental principles with a fairly detailed survey of modern practical developments. This is just such a book, and it is a successful one. The outlook is that of the technician rather than the natural philosopher, but the treatment is thorough as far as it goes, and the book will be of value to all experimentalists who have occasion to employ electrical technique, and to students whose interests lie on the practical side of the subject.

Since the book is primarily an exposition of the subject as an experimental science, practical units are used consistently throughout. There can be little doubt that experimentalists will find this procedure preferable to the traditional one of employing different systems for electrostatic and electromagnetic phenomena, with a third for practical purposes. The author's treatment shows clearly that the use of a consistent absolute system of practical units does not complicate the equations to any noticeable extent: indeed with the ampere-turn per metre as the unit of H, i.e. a rational system, there is on the whole a net gain in simplicity. It must be confessed that the magnetic units of the system are at present unfamiliar quantities, and that magnetization curves showing values of magnetic induction in Webers per square metre are apt to look somewhat strange. Nevertheless the International Electrotechnical Commission has recommended the use of this unit, in the interests of consistency and simplicity, and if their recommendation is acted upon in all the countries represented on the Commission, the unfamiliarity will soon disappear.

As regards scope, the book gives a systematic account of the phenomena of electrostatics and dielectrics, of direct and alternating currents, of electromagnetism and the magnetic properties of matter, electrolytic, thermal, and photoelectric effects, all on more or less usual lines. In addition, there is a much more detailed treatment than is usual of those parts of the subject which have led to important technical developments, for instance non-linear circuit elements including rectifiers, thermionic vacuum tubes, filters and coupled circuits, amplifiers, oscillators, antennae and radiation. In short, the author may be said to have gone for his illustrations to the modern research laboratory instead of to the scientific museum, and to a large class of students his book will be the more welcome on that account.

L. H.

Modern Magnetism, by L. F. Bates. Pp. x+340. (Cambridge: at the University Press, 1939.) 16*s*.

For its effective influence on future progress, scientific research, with its voluminous primary literature, is increasingly dependent on good textbooks. Magnetism was for so long badly served, as a very poor relation of electricity, that, although a number of books dealing with various aspects of the subject have appeared in the last few years, the ground which can be usefully covered is far from exhausted. A new book on magnetism is, therefore, most welcome, particularly as the author has weighted his treatment in accordance with good judgment as to what has previously been lacking. He aims at removing some of the difficulties which he believes the average student to find, and which he attributes to the subject usually being treated too much from a theoretical standpoint. Whether fuller accounts of experimental work will help in removing theoretical difficulties is open to question, but undoubtedly the accounts included here will serve a most useful purpose.

The first chapter on "Fundamental conceptions" covers rapidly a very wide range—elementary definitions, the classical theory of diamagnetism and paramagnetism, the vector-model treatment of the atom, and the quantum theory of susceptibility. The next two chapters deal with the production and measurement of fields, and methods for the measurement of susceptibility. There are full details of the various standard methods, with useful notes and comments suggested by practical experience, and adequate accounts are given of more specialized apparatus. Illustrative experimental results are briefly discussed. An account is also given of single-crystal measurements. A description of the fundamental experiments on the magnetic deviation of atomic rays is followed by an account of the recent remarkable work on nuclear moments. The admirable critical survey of the various experiments on the gyromagnetic effect would be expected from one of the pioneer workers in this field.

The last part of the book deals mainly with ferromagnetism. The chapter on magnetic saturation and the equation of state includes good accounts of the methods for measuring magnetization in high fields, and for the determination of the Curie point. In the treatment of magnetothermal effects, some of the recent theoretical work on the problem of the molecular field in ferromagnetics is discussed, and a fairly full description is given of the adiabatic demagnetization technique for the production of low temperatures. The final chapter deals with magnetostriction and, more briefly, the phenomena of the Barkhausen effect.

The reviewer would be loath to cast the first stone at an author for peculiarities in the treatment of magnetic energy, or for occasional errors—these are challenges to the reader, who may also wonder why a recumbent script symbol was chosen to represent magnetic field. The book as a whole suffers to some extent from the fact that experimental results and discussions of them are for the most part given as incidental to the description of the experimental methods. As a consequence, for those less familiar with the general field, the interrelations between the parts may be somewhat obscured. The impression given is that the phenomena under investigation arose from the technique, rather than that the technique was designed to investigate the phenomena. This, however, is of minor importance when the book is expressly designed to redress a balance. The author, with his considerable inside knowledge of diverse magnetic techniques, deals admirably with the experimental side of the subject, provides adequate links with other treatments, and also discusses some of the very recent theory. The book is one which should be in the hands of all who are concerned with magnetism. To those embarking on experimental work on the subject it will be invaluable.

E. C. S.

An Introduction to Electrical Engineering, by Prof. E. W. MARCHANT, D.Sc., F.G.G.I. Pp. xii + 297, 8½ × 5½ in. 220 diagrams. (London: Methuen and Co., 1939.) 12*s*. 6*d*. net.

Prof. Marchant has written this textbook for students taking first-year and second-year courses in electrical engineering at a university or technical college. In contrast with textbooks intended for the student of physics, as one expects, it presents the principles more briefly and gives more prominence to practical applications. Of the 29 chapters of the present book 8 are concerned with dynamos and motors and 7 with alternating currents and their applications. It is interesting to observe the extent to which a line of application can be pursued (as, for example, in chapter XXVII on the transformer), with the advantageous result that the principle is driven well home and the idea of design comes into view.

A striking feature of the book is the introduction of the m.k.s. (metre-kilogram-second) system of units. It is alluring to discover that the practical units associated with the c.g.s. system, namely, the volt, ampere, ohm, etc., turn out to be the absolute units on the m.k.s. system. But, on the other hand, it must be a somewhat difficult task for the teacher to explain the choice, on this system, of a fictitious medium of unit dielectric constant which on the c.g.s. system would have the value 3×10^9 for its dielectric constant, and similarly of a medium of unit permeability which on the c.g.s. system would have to have the value 10^7 assigned to it.

The presentation of the subject matter is generally clear and interesting, and the ground covered is adequate.

A few omissions and errors of statement may however be pointed out. The introductory description of the structure of atomic nuclei is not quite up to date. In defining the international units of current, potential and resistance attention might have been directed to the decision to abandon these units as from the beginning of next year, 1940. On p. 39 the impression is given that *all* metals become superconductors at very low temperatures. On p. 57 is the somewhat loose statement "the practical unit of e.m.f. is the standard cell". The formula for mercurous sulphate is wrongly given on p. 57, and the calorie is wrongly defined on p. 60. The "kVA." is introduced on p. 219 without definition. Space might have been found for a description of the cathode-ray oscillograph, which receives only a bare mention. One further criticism: it is most desirable that a clear notation should be adopted to distinguish instantaneous value, amplitude, root-mean-square value and vector value of an alternating quantity such as current; more care might have been expected in this respect in Chapter XXII and the succeeding chapters.

A useful series of examples is given at the close of each chapter, and answers will be found at the end of the book.

D. O.

Alternating Current Bridge Methods, by B. HAGUE. Fourth edition. Pp. 587. (London: Pitman. 1938.) 25*s*. net.

In preparing a fourth edition of his well-known book on alternating-current bridges, Dr B. Hague has made an important contribution to the literature of the subject, for clearly he has made a careful survey of the vast number of papers bearing on bridge measurements that have been published in recent years, and has thoroughly incorporated all the new matter into the general scheme of the book. The result is that not only has he provided the specialist with a very convenient classified and indexed summary of all the recent research in this direction, but also the book provides the student with a systematic account of the whole subject from its early historical development to the most recent improvements in the details of technique. The treatment is throughout full and clear, and the volume is particularly valuable as a work of reference. So much new matter has been added that the book is now fairly bulky, and from the point of view of the student it might have been an advantage if some of the older material had been condensed. However, the detail is probably necessary for reference purposes, and the use of type of two sizes gives the reader some idea as to the relative importance of the various parts.

Bridge circuits are now so numerous that even the experienced worker may be forgiven if he finds their tale a little overwhelming in spite of a systematic classification. However, a chapter on the choice of a bridge method serves as an admirable guide, indicating which of the preceding methods should be consulted by those wishing to make measurements of various specified quantities. Dr Hague's advice is usually good. It is only occasionally that one wants to contradict him violently. Should he, for example, say that when a telephone receiver is used as detector "it is best to support it in a stand and listen through stethoscope ear tubes"? This procedure is surely "best" on paper only. There is also a statement to the effect that the dielectric constant of a conducting liquid "can be found at once from the effective capacitance of an electrolytic cell". A reader who acted on this advice might go very much astray, for the effective capacitance of such a cell is apt to be almost entirely determined by the polarization capacity of the electrodes and the resistance of the liquid. One other complaint arises from a curious omission on the historical side: the decade constant-inductance resistor appears to be attributed to an American invention of 1934, but such resistors were devised by Albert Campbell and made by R. W. Paul more than twenty years earlier. The constant-inductance slide wire is correctly attributed to Mr Campbell, but his decade resistor appears to have been overlooked. These are, however, only minor matters, and the book as a whole can be unreservedly recommended.

L. H.

Theoretical Hydrodynamics, by L. M. MILNE-THOMSON. Pp. xxiv + 552. (London: Macmillan. 1938.) With four plates. 31*s*. 6*d*.

The need for a book on hydrodynamics presenting the subject from a modern angle has long been felt, and Prof. Milne-Thomson is to be congratulated on having succeeded in writing one which, I venture to say, will take its place alongside Lamb's treatise as a standard work on the subject.

In this work the author departs radically from the old mode of presentation of the subject: it begins with a series of interesting plates showing the flow of a fluid past a fixed cylinder and a fixed aerofoil. These are followed by a chapter (I) of an introductory nature on fluid motion. Vectors are used whenever this is possible and a separate chapter (II) has therefore been devoted to the study of vectors. I think that it must be difficult to find anywhere else such a concise and comprehensive account of the essential portions of vector analysis ordinarily required for the study of hydrodynamics, or any other subject which lends itself to vector treatment.

The general properties of fluid motion, the derivation of the equation of motion and of the equation of energy, and the general characteristic of vortex motion are all dealt with in chapter III. The eleven following chapters (IV to XIV inclusive) are devoted to problems in two dimensions, chapter XIV being devoted to wave motion. The use of conformal transformations in solving problems in two-dimensional fluid motion has been fully dealt with, and in a systematic way, and the results are applied, as far as possible, to various aerodynamical problems. A whole chapter (V) on complex numbers regarded as linear operators has been added and in it the theory of functions of a complex variable is developed as far as it is required for the subsequent theory. Like the chapter on vectors, it is extremely well written and interesting to read.

The remaining chapters in the book deal with streaming motion in three dimensions, the motion of a solid through a perfect fluid, and vortex motion. At the end of this chapter a short account is given of the Prandtl theory of an aerofoil of finite span. It is only in the last chapter that viscosity is considered, and it is with some disappointment that one then finds only about forty pages devoted to the subject. Much has been condensed in this chapter, including such topics as the flow of heat in a viscous fluid in motion, and Filon's formula for the lift and drag on a stationary cylinder in a stream of viscous fluid. But too little space has been given to such important subjects as the theory of the boundary layer and G. I. Taylor's work on the stability of a viscous fluid between two rotating cylinders, whilst some mention should have been made of Osborne Reynold's work on turbulence, of the theory of eddy viscosity, and of the theory of compressible fluids. This is the only criticism which must be made of a book which should be considered seriously by every worker on the subject. For the student an attractive feature will be the set of exercises at the end of every chapter (numbering about 500 in all), many of which have been chosen from the papers set in the Cambridge Mathematical Tripos and the London M.Sc. examination.

The book is beautifully printed and the diagrams, which number about 330 in all, are very clear and extremely well drawn.

V. C. A. F.

Kinetic Theory of Gases, by EARLE H. KENNARD. Pp. xiii+483. (International Series in Physics. London: McGraw-Hill Publishing Co., Ltd.) 30*s*.

To many the appearance of a new treatise on the kinetic theory of gases may come as something of a surprise, but whoever opens this volume is speedily assured that the subject has advanced considerably since the standard works of Jeans and Loeb were written. Indeed, although the author has made no attempt to formulate a complete bibliography, reference is made to nearly one hundred papers published during the past ten years.

The first eight chapters deal mainly with the normal aspects of the classical theory of gases, but use is frequently made of the principles of statistical mechanics. These chapters cover such subjects as the distribution law for molecular velocities, the general motion and spacial distribution of the molecules, the viscosity, thermal conduction and diffusion of gases, the equation of state, energy, entropy and specific heats, Brownian motion and the properties of highly rarefied gases. The ninth chapter commences with an account of statistical mechanics in so far as it relates to the methods used in the earlier chapters, and then goes on to consider how the statistical investigation is to be conducted when wave mechanics is substituted for the classical theory. In the next chapter the theory of gases is developed entirely on wave-mechanical lines, treatments being given for both the Fermi-Dirac and Bose-Einstein types of point-mass gas. The book ends with a short account of the electric and magnetic properties of gases.

A few problems are interspersed throughout the text. These appear to have been chosen with a view to amplifying the scope of the book and at the same time testing the reader's appreciation of the methods involved.

The subject matter is presented in a clear, attractive manner, and the proofs appear to have been checked with care. The book is certainly up to the standard which one has come to associate with the International Series in Physics, and is a work to be recommended not only as a first class textbook for the student, but also as a helpful guide for the more advanced worker, whether seeking to avail himself of existing knowledge in this field or planning to extend such knowledge.

R. W. P.

An Introduction to Vector Analysis for Physicists and Engineers, by B. HAGUE, D.Sc., F.C.G.I. Pp. viii + 118. (London: Methuen and Co., Ltd. 1939.) 3*s.* net.

This recent addition to Methuen's series of monographs on physical subjects provides a contribution to mathematical physics which the special student will be quick to appreciate. Once familiarity with the notation of vector analysis has been acquired, the power of its methods will soon yield fruit in the fields of hydromechanics and electromagnetic theory. For example, the derivation of Poynting's formula for the rate of transfer of energy in the electromagnetic field is reduced to a matter of a few lines.

The subject is here developed *ab initio*, and the applications to electric potential and electric and magnetic fields, conceptions with which the student will be already familiar, make it easy to follow the course of this development.

In a final chapter tensors and their matrices, and the dyadic, are briefly introduced, and the student has a glimpse at least of the way to other fields, for instance in the theory of relativity. Brief appendices deal respectively with the use of polar co-ordinates as an alternative to rectangular Cartesian co-ordinates, and with a brief bibliography of vector analysis.

D. O.

The Theory and Use of the Complex Variable, by S. L. GREEN. Pp. viii + 136. (London: Sir Isaac Pitman and Sons, Ltd.) 10*s.* 6*d.*

The applied mathematician uses the complex variable in several ways, and in particular for the evaluation of real definite integrals, for shortening trigonometrical expressions and for effecting conformal transformations from a more to a less complicated region. This short book, starting with the elementary notions of complex numbers and their vectorial representation, gives an introduction to all these uses, and shows them in actual use in the last two chapters, devoted to potential problems and to alternating-current theory respectively.

After the introductory chapter follow two on de Moivre's theorem, and the familiar bookwork associated with it, and on infinite series. Although the book is only an introduction, a student who works through it, and solves exercises like the following, will have had a very good grounding.

(1) If α is a complex root of $z^{13}=1$, prove that $\alpha+\alpha^5+\alpha^8+\alpha^{12}$ is a root of

$$z^3+z^2-4z+1=0.$$

(2) $$\prod_{s=0}^{n-1}[a\cos(2s\pi/n)+b\sin(2s\pi/n)-c]=(-\tfrac{1}{2})^{n-1}r^n(\cos n\phi-\cos n\theta)$$

where $r=+\sqrt{(a^2+b^2)}$, $a=r\cos\phi$, $b=r\sin\phi$, $c=r\cos\theta$ and $(a^2+b^2)>c^2$.

The chapter on the general theory of functions of a complex variable, with its account of Cauchy's theorem and the doctrine of residues, is very clear. It might be well in a future edition to point out explicitly how real definite integrals can be deduced from most contour integrals.

There are two chapters devoted to conformal representation, and, as has already been noted, two on the applications of the subject.

The book concludes with an appendix suggesting the best future reading for various types of student, according to their several interests, and is one which may be recommended not only for mathematical students at an early stage, but also for those who in later life wish to renew their acquaintance with this subject, or to fill a gap which may have been left in earlier reading.

J. H. A.

The Phase Rule and its Applications, by A. FINDLAY, 8th edition. Pp. xv + 327. (London, New York, Toronto: Longmans Green and Co.) 12*s*. 6*d*. net.

When a book reaches its eighth edition it goes without saying that it is a good book, or at least has been a good book. The reviewer's remarks will therefore be confined to the question whether the book is up to date. A rough count shows that the number of references to the literature of the last twenty years is about a third of the number for the previous period of twenty years. This gives the erroneous impression that the subject, if not dead, is at least dying fast. The description of the use of x rays is, apart from a few isolated sentences, confined to two pages. There is no mention of isotopes, not even deuterium; none of parahydrogen and orthohydrogen; none of helium; none of lambda points; none of order-disorder changes in alloys; none of metal-hydrogen systems except palladium-hydrogen. The discussion of the palladium-hydrogen system is out of date by fifteen years, the excellent measurements of Gillespie and their interpretation being left unmentioned. It is to be hoped that in the ninth edition an adequate account will be given of at least some of these subjects of current interest, so as to restore the former very high standard of this still popular book.

E. A. G.

Intermediate Chemistry, by (the late) T. M. LOWRY and A. C. CAVELL, 2nd edition. Pp. xvi + 876. (London: Macmillan.) 12*s*. 6*d*.

In this book an attempt has been made to provide within one pair of covers for the entire needs of the student who is working for his intermediate examination. A great deal of care has obviously been expended in the effort to ensure that the volume shall be a worthy companion at all times: problem papers are given with brief solutions, as well as logarithm tables for working them out. The book is well printed, and has waterproofed covers which will withstand laboratory conditions better than most types of binding.

The main part of the book is divided into six sections, of which the first three deal with inorganic chemistry and the others with qualitative and quantitative analysis, physical chemistry, and organic chemistry respectively.

It is in the inorganic section that the book differs most from those which were popular twenty years ago. Quite near the beginning, chemical combination is explained in terms either of electron transfer or of electron sharing, and an acid is defined as a proton-donor, a base as a proton-acceptor. The whole treatment is based on this logical outlook, the student being familiarized with such ideas as that high melting points and densities are associated with electrovalency and low ones with covalency. The order in which compounds are described is determined rather by their metallic than their acid constituents, and the elements themselves are related rather to Bohr's periodic table than to earlier models.

The peculiarities of the transition elements and the rare earths are explained by the expansion of an inner shell of electrons before an outer shell resumes its growth, so that the student finds them much less puzzling than did earlier generations. In connexion with the inorganic section, it might be suggested that the minerals so often mentioned as sources of metals might all be included in the index of any future edition. They are unsystematic, and a student who has forgotten what diaspore is has no immediate means of finding out.

The section on physical chemistry is limited by the unstated assumptions that the student knows no calculus and no thermodynamics, but within the limits thus set, an excellent descriptive account of electrochemistry, critical phenomena and chemical kinetics is given.

As regards the section on organic chemistry, a reviewer who is ignorant of the subject can only say that he found it interesting and that it appears to be a sound introduction to the subject. The authors keep in mind the needs of those students who will carry their study no further, and who must therefore learn now, if ever, something of the modern drugs and dyestuffs. In this section, the utility of physics in reducing chemistry to a logical structure is again exemplified in many places.

J. H. A.

Theoretical Electrochemistry, by N. A. McKenna. Pp. xiii+469. (London: Macmillan and Co. 1939.) 15*s*.

After a short historical introduction and a good chapter on fundamental electrical measurements, the author discusses the modern theory of conductivity, transport numbers, ionic mobilities, viscosities, diffusion, and the compressibilities and surface tensions of strong electrolytes. Many equations are stated without deduction. The treatment of chemical thermodynamics which follows is not very satisfactory, for instance the sign of the work equation at the top of p. 153 is wrong, the author often speaks of a "system" when he means a "change", and something has gone wrong with entropy on p. 153. This chapter requires overhauling. The discussion of the Debye-Hückel equation in chapter VII is unnecessarily long and difficult, and could have been simplified with a little trouble. The remaining chapters deal with electrode processes, reversible cells, irreversible electrode processes, ions in solution, ionic equilibria in acid and base solutions, hydrolysis, neutralization and the theory of indicators. Although experimental methods are frequently given (a good feature of the book) the descriptions are not always sufficiently clear, a notable example being the chapter on transport numbers in which the fundamental equation on p. 117 has a ratio inverted. Some names of authors are incorrectly given, and the references are mostly to rather old work. No new work on oxidation-reduction potentials and polybasic acids, for example, is mentioned, and these sections are not up to date. In spite of its faults the book has distinct merits and contains some information not previously collected in one volume. It will be found useful by students who wish to get up the subject of modern electrochemistry, but they must supplement it by reading some modern work in various sections.

J. R. P.

The Principles of Electrochemistry, by D. A. MacInnes. Pp. 478. (London: Chapman and Hall Ltd.; New York: Reinhold Publishing Corporation. 1939.) 30*s*. net.

In his preface, the author says his book "has been written with the idea of furnishing an account of theoretical electrochemistry as it is today, and to satisfy an inner urge to see the subject he is interested in as a logical, connected whole". The result is very good.

The outlook is that of the laboratory rather than the writing desk, so that although the theory of the subject is developed in a very satisfactory manner, with the use of a clean system of symbols, the way in which experimental results have been obtained and calculated is always to the front, and the numerous illustrations bring out very well the degree of accuracy and the agreement with theory. A good feature is the inclusion of numerical calculations when these are the best way of explaining a subject, as in the chapter on transport numbers. The sound judgment of the author is often in evidence, as in the appreciation of the third law of thermodynamics on p. 123 and his apology for the "multitude of definitions" in the section on activity coefficients (p. 131). Many touches, such as the note on the wider meaning of chemical potential on p. 125, indicate the wide knowledge of the author.

Among the many good things in the book, the careful and clear deduction of the Debye and Hückel equation in chapter 7, the neat deduction of the equation for the e.m.f. of a cell with transport on p. 157, and the revision of the results for conductivities in alcoholic solutions (which show that the Debye and Hückel equation holds up to $0{\cdot}06N$) may be mentioned. The chapters are on the history of electrochemistry, Faraday's law (only one is stated), conductance, transport, thermodynamics, the Debye-Hückel equation, concentration cells, standard potentials, non-aqueous solutions, cells with liquid junctions, *p*H values, oxidation-reduction, potentiometric titrations, interionic attraction, theory of conductance and its consequences, conductance in mixed solvents, ionization of organic acids and bases, dipole moments, electrokinetic phenomena, passivity and overvoltage, and other less important subjects. Much important work in some of these fields has been carried out by the author, and his book may be warmly recommended as a thoroughly sound and modern presentation of the subject, written very clearly and with excellent balance.

J. R. P.

Stephen Timoshenko, 60th Anniversary Volume. Pp. viii + 277. (London: Macmillan and Co., Ltd.) 22*s.*

A book comprising twenty-eight separate papers, by as many different authors, on problems covering the whole field of engineering research from analysis of plastic strain in a cubic crystal to transient torques in induction-motor drives presents an almost impossible task to the reviewer. The contributions to this new type of omnibus volume have been prepared in honour of Prof. Stephen Timoshenko on the occasion of his sixtieth birthday. Naturally many of the papers deal with problems in those fields in which Prof. Timoshenko himself has made so many notable contributions, although the inclusion of only two papers on stability is a little surprising, until one remembers how little scope for further research Timoshenko has left in this branch. The magnitude of Timoshenko's contribution to the theory of elasticity, particularly in relation to stability and problems of vibration, is shown by the list of his fifteen books and seventy-one papers, in Russian, German, French and English, with which Mr Lessells concludes a brief account of Timoshenko's career.

In addition to the two papers on stability, the book includes six papers on problems of vibration, ten on problems of elasticity, six on plasticity and creep, two on fatigue, and one each on the theory of solid friction and the effect of recrystallization on mechanical properties of copper. The list of contributors includes many of the names most familiar in these fields and, although the majority are American, the European contribution is reasonably representative.

The book is most attractively printed and bound, and the illustrations are excellently reproduced. Perhaps the chief value of the book, apart from its sentimental appeal to Prof. Timoshenko's many admirers, may be as an epitome of the state of engineering research in 1938.

H. L. C.